TWO-DIMENSIONAL THERMAL STRESS IN A

MULTIPLY CONNECTED PLATE

A THESIS

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MULTIPLY CONNECTED PLATE

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SUMMARY

Two methods of determining thermal stresses in a multiply connected body are investigated and evaluated. One method involves the solution of the governing equations for this type of problem using finite difference approximations. The other method utilizes the dislocation analogy of Biot to solve for the thermal stresses photoelastically. The development of each method is given and explained in detail.

To carry out the investigation of the two methods a particular problem was chosen to be solved by each method. A square flat plate with a center circular hole was the geometrical configuration that was used. The selected temperature distribution was assumed to be steady state and to satisfy Laplace's equation.

The results obtained by the two methods agreed quite closely. These results indicated that both methods were practical and yielded reliable results. It was found that to determine the thermal stresses at a few isolated points in a body quickly the photoelastic method was the best if the necessary equipment was available. However, if a stress distribution over the entire body was needed, then the numerical method was the preferred method.

CHAPTER I

INTRODUCTION

Thermal stresses are stresses which are induced when a body that is subjected to temperature changes is restrained from contracting or expanding. These stresses may result from external restraints such as preventing a body from expanding when it is heated, or they may also result from internal restraints. A temperature gradient within a body may cause thermal stresses.

It can be shown that for a general simply connected body, for which a plane stress or plane strain analysis can be used to calculate the stress distribution, a temperature distribution which satisfies Laplace's equation does not produce any thermal stresses. This assumes that the material constants do not vary with temperature. However, for a similar body which is multiply connected the presence of the same type of temperature distribution can induce thermal stresses.

It is the object of this study to investigate and evaluate two methods of determining the two-dimensional thermal stresses in a multiply connected body with a temperature distribution that satisfies Laplace's equation. This investigation is carried out for plane bodies, but the results are easily adapted to long cylindrical bodies.

One of the methods used is analytical in nature and the other is experimental. The analytical method involves the solution of the governing equations by finite difference approximations. The experimental method is the application of the dislocation analogy to a photoelastic model from which stresses that are equal and opposite to the thermal stresses are determined.

The Particular Problem

The particular problem is the determination of the two dimensional plane elastic stresses in a flat square plate with a center circular hole. The dimensions of the plate are 6 in. by 6 in., and the diameter of the hole is 3 in. There is no external loading on the plate. All stresses are due entirely to the two-dimensional temperature distribution. The temperature distribution that is assumed for the plate is steady state with the inside boundary at 100 degrees F and the outside boundary at 0 degrees F. There is no internal heat source or sink. All material constants used are those for steel.

CHAPTER II

MATHEMATICAL CONSIDERATIONS

The mathematical model of the general two-dimensional plane stress problem is the so-called free slice. The free slice has cylindrical boundaries, and the thickness is small in comparison to the other dimensions. Also the two parallel bounding planes must be unstressed and free to warp, and the loading must be distributed symmetrically across the thickness so that the middle plane remains plane. Under these conditions the state of stress given by

$$\sigma_z = \tau_{yz} = \tau_{xz} = 0$$

is obtained to a close approximation. The xy plane is the middle plane of the slice.

The equilibrium conditions for internal stress for a two-dimensional problem of this type, with no body forces, are

$$\frac{\partial \sigma_x}{\partial_x} + \frac{\partial \tau_{xy}}{\partial_y} = 0 \tag{1}$$

$$\frac{\partial \sigma_y}{\partial_y} + \frac{\partial \tau_{xy}}{\partial_x} = 0$$

The strain-displacement-stress-temperature relations for a free slice are

$$\epsilon_x = \frac{\partial u}{\partial x} = \frac{1}{E}\left(\sigma_x - \mu\sigma_y\right) + \alpha T \tag{2}$$

$$\epsilon_y = \frac{\partial v}{\partial y} = \frac{1}{E}(\sigma_y - \mu\sigma_x) + \alpha T$$

$$\epsilon_z = \frac{\partial w}{\partial z} = -\frac{\mu}{E}(\sigma_x + \sigma_y) + \alpha T$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{2(1 + \mu)\tau_{xy}}{E}$$

In the absence of body forces the stresses may be expressed in terms of Airy's stress function ϕ as follows.

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} \tag{3}$$

$$\sigma_y = \frac{\partial^2 \phi}{\partial x^2}$$

$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

It is easily seen that these relations satisfy the equilibrium equations identically.

The compatibility equations for a plane stress problem are

$$\frac{\partial^2 \epsilon_y}{\partial z^2} + \frac{\partial^2 \epsilon_z}{\partial y^2} = 0 \quad , \quad \frac{2\partial^2 \epsilon_x}{\partial z \partial y} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial z} \tag{4}$$

$$\frac{\partial^2 \epsilon_z}{\partial x^2} + \frac{\partial^2 \epsilon_x}{\partial z^2} = 0 \quad , \quad \frac{2\partial^2 \epsilon_y}{\partial z \partial x} = \frac{\partial^2 \gamma_{xy}}{\partial y \partial z}$$

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \; , \quad \frac{2\partial^2 \epsilon_z}{\partial x \partial y} = -\frac{\partial^2 \gamma_{xy}}{\partial z^2}$$

In general for plane stress all of these relations restrict the function ϕ. However in the absence of body forces the Z dependence of ϕ vanishes and only the condition

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \tag{5}$$

need be used[3]

Using the stress-strain-temperature relations and the stress function ϕ the compatibility equation becomes

$$\frac{\partial^4 \phi}{\partial x^4} + \frac{2\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = - E\alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \tag{6}$$

This is the governing equation for the stress function ϕ. For steady state heat transfer where there is no heat source term, the temperature distribution satisfies Laplace's equation;[13] therefore, the compatibility equation becomes

$$\frac{\partial^4 \phi}{\partial x^4} + \frac{2\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0 \tag{7}$$

For a simply connected region this equation provides the necessary and sufficient condition for the displacements u, v, and the rotation ω to be single-valued. For a multiply connected region the situation is somewhat different in that the equation provides a necessary but not a sufficient condition. See Appendix A.

Displacements in a Multiply Connected Slice

To insure that u, v, and ω are single-valued in a multiply connected slice the following conditions must be satisfied:

$$\int_{C_i} d\omega = 0 \tag{8}$$

$$\int_{C_i} du = 0$$

$$\int_{C_i} dv = 0$$

where C_i is a curve that completely encircles any interval boundary. These conditions must be satisfied around every internal boundary in the slice.

Using these conditions along with the compatibility equation, the stress-strain-temperature relations and Airy's stress function the following integral equations are obtained.[4]

$$\int_{C_i} \frac{\partial(\nabla^2\phi)}{\partial n}\, ds = -E\alpha \int_{C_i} \frac{\partial T}{\partial n}\, ds \tag{9}$$

$$\int_{C_i} \left(y\, \frac{\partial(\nabla^2\phi)}{\partial n} - x\, \frac{\partial(\nabla^2\phi)}{\partial s} \right) ds = -E\alpha \int_{C_i} \left(y\, \frac{\partial T}{\partial n} - x\, \frac{\partial T}{\partial s} \right) ds$$

$$\int_{C_i} \left(y\, \frac{\partial(\nabla^2\phi)}{\partial s} + x\, \frac{\partial(\nabla^2\phi)}{\partial n} \right) ds = -E\alpha \int_{C_i} \left(y\, \frac{\partial T}{\partial s} + x\, \frac{\partial T}{\partial n} \right) ds$$

where

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}$$

These equations are usually referred to as Michell's equations.

From the above it is seen that a function ϕ which can be used to determine the complete stress distribution must satisfy the equation

$$\frac{\partial^4\phi}{\partial x^4} + \frac{2\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4} = 0 \tag{7}$$

and Michell's equations. Also ϕ must satisfy certain boundary conditions at all of the boundaries.

Boundary Conditions of ϕ

At a point on a boundary the function ϕ must satisfy the equations

$$\frac{\partial^2\phi}{\partial y^2}\frac{dy}{ds} + \frac{\partial^2\phi}{\partial x\partial y}\frac{dx}{ds} = \overline{X} \tag{10}$$

$$\frac{\partial^2\phi}{\partial x^2}\frac{dx}{ds} + \frac{\partial^2\phi}{\partial x\partial y}\frac{dy}{ds} = \overline{Y}$$

where $\overline{X}$ and $\overline{Y}$ are the x and y components of boundary loading.[4] These conditions must be satisfied at all points on all boundaries. For boundaries that are free from external loading $\overline{X} = \overline{Y} = 0$. Then the boundary conditions become

$$\frac{d}{ds}\frac{\partial\phi}{\partial y} = 0 \tag{11}$$

$$\frac{d}{ds}\frac{\partial\phi}{\partial x} = 0$$

on each load free boundary. Integration of these equations yields

$$\frac{\partial\phi}{\partial x} = a, \quad \frac{\partial\phi}{\partial y} = b$$

Now

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy$$

$$= adx + bdy$$

Integration of this equation gives

$$\phi = ax + by + c \tag{12}$$

where a, b, and c are called boundary constants. The function must be of this form on each boundary. For a region with two boundaries ϕ takes the form

$$\phi = a_1 x + b_1 y + c_1 \quad \text{on one boundary}$$

and

$$\phi = a_2 x + b_2 y + c_2 \quad \text{on the other boundary}$$

Determination of Boundary Constants

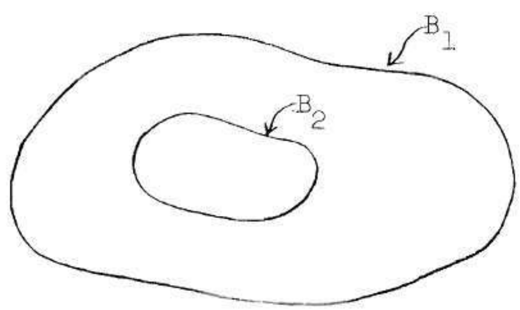
Prager[2] has developed a method of using special solutions to determine the a's, b's, and c's for a doubly connected domain at uniform temperature. Holms[12] extended the method of Prager to regions with more than one hole and with temperature distributions present.

The solution function ϕ is taken as a linear combination of six particular solutions, that is

$$\begin{aligned} \phi_1 = \; & a_1 \phi_{A1} + b_1 \phi_{B1} + c_1 \phi_{C1} \\ & + a_2 \phi_{A2} + b_2 \phi_{B2} + c_2 \phi_{C2} \end{aligned}$$

for a region with one internal boundary. The boundary conditions for $\phi_{A1}, \phi_{B1}, \ldots, \phi_{C2}$ must be selected so that the complete solution will satisfy the boundary conditions on all boundaries. Since the stresses are given by the second derivative of ϕ, the addition of a linear function of the coordinates to ϕ will leave the stresses unaltered. The assumption is made that this addition is accomplished so that the boundary constants are zero on one of the boundaries. For the particular problem at hand the boundary constants for the internal boundary are taken as zero. Then the problem is reduced to finding one set of boundary constants a_1, b_1, and c_1. The method for accomplishing this is outlined in the following.

Consider a region with one internal boundary.



The stress function ϕ for this region is given by

$$\phi = a\phi_A + b\phi_B + c\phi_C \tag{13}$$

where

1. ϕ_A is a solution to the equation

$$\nabla^4 \phi_A = 0$$

with boundary conditions.

a. $\phi_A = \frac{\partial \phi_A}{\partial x} = \frac{\partial \phi_A}{\partial y} = 0$ on B_2

b. $\phi_A = x, \quad \frac{\partial \phi_A}{\partial x} = 1, \quad \frac{\partial \phi_A}{\partial y} = 0$ on B_1

2. ϕ_B is a solution to the equation

$$\nabla^4 \phi_B = 0$$

with boundary conditions

a. $\phi_B = \frac{\partial \phi_B}{\partial x} = \frac{\partial \phi_B}{\partial y} = 0$ on B_2

b. $\phi_B = y, \ \frac{\partial \phi_B}{\partial x} = 0, \ \frac{\partial \phi_B}{\partial y} = 1$ on B_1

3. ϕ_C is a solution to the equation

$$\nabla^4 \phi_C = 0$$

with boundary conditions

a. $\phi_C = \frac{\partial \phi_C}{\partial x} = \frac{\partial \phi_C}{\partial y} = 0$ on B_2

b. ${}_C = 1, \ \frac{\partial \phi_C}{\partial x} = \frac{\partial \phi_C}{\partial y} = 0$ on B_1.

From these conditions particular values for ϕ_A, ϕ_B, and ϕ_C can be obtained. Then the function

$$\phi = a\phi_A + b\phi_B + c\phi_C \tag{13}$$

can be substituted into the integral equations (9). This gives a system of 3 equations with three unknowns, a, b, and c. This system can be solved for a, b, and c. It is seen then that ϕ is the required solution, since this total ϕ satisfies the differential equation (7), satisfies the integral equations (9), and has the proper values at the boundaries as given by equation (12).

This method can be extended to the case where the temperature distribution is not steady state by letting

$$\phi = a\phi_A + b\phi_B + c\phi_C + \phi_D$$

ϕ_A, ϕ_B, and ϕ_C are obtained as before. ϕ_D is the solution to the equation

$$\nabla^4 \phi_D = - E\alpha \nabla^2 T$$

with boundary conditions

$$\phi_D = \frac{\partial \phi_D}{\partial x} = \frac{\partial \phi_D}{\partial y} = 0 \quad \textit{on} \quad B_1 \quad \textit{and} \quad B_2$$

The constants a, b, and c are obtained by the same method as for the steady state case.

CHAPTER III

THE METHOD OF SOLUTION

The solution of two-dimensional thermal stress problems of the type being considered requires the solution of the homogeneous biharmonic equation three times for the stress functions ϕ_A, ϕ_B, and ϕ_C. When the solutions are found a function ϕ given by

$$\phi = a\phi_A + b\phi_B + c\phi_C$$

is formed. This function ϕ must be substituted in the integral equations (9) to yield values for the boundary constants a, b, and c. When this is accomplished the function ϕ may be used to calculate the stresses from equations (3).

To carry out the steps of the particular problem finite difference approximations were used. The biharmonic equation was solved using a numerical relaxation procedure. The integral equations were evaluated using Simpson's Rule and the stresses were calculated by numerical differentiation.

Finite Difference Formulation

In order to use the finite difference approximations a rectangular array of grid points are superimposed on the upper half of the plate. The symmetrical and antisymmetrical properties of the stress functions and the temperature distribution allow the calculations for the lower half of the plate to be carried out in terms of the upper half. The grid spacing h has been set equal to 0.25 in. Indices i and j are used to locate the grid points. See Figure 1. Using the notation

$$\nabla x^2 \phi_{ij} = \phi_{i+1,j} - 2\phi_{ij} + \phi_{i-1,j} \tag{14}$$

$$\nabla y^2 \phi_{ij} = \phi_{i,j+1} - 2\phi_{ij} + \phi_{i,j-1}$$

and

$$\nabla x^4 \phi_{ij} = \phi_{i+2,j} - 4\phi_{i+1,j} + 6\phi_{ij} - 4\phi_{i-1,j} + \phi_{i-2,j} \tag{15}$$

$$\nabla y^4 \phi_{ij} = \phi_{i,j+2} - 4\phi_{i,j+1} + 6\phi_{ij} - 4\phi_{i,j-1} + \phi_{i,j-2}$$

the finite difference form of the biharmonic equation is

$$\nabla x^4 \phi_{ij} + 2\nabla x^2 \nabla y^2 \phi_{ij} + \nabla y^4 \phi_{ij} = 0 \tag{16}$$

The boundary conditions which are used to solve the above equation depend on whether the equation is being solved for ϕ_A, ϕ_B, or ϕ_C.

On the outside boundary the boundary and the grid points coincide, however since the inside boundary is curved the grid points do not necessarily fall on the boundary. In order to try and obtain the best approximation to the boundary conditions on the curved boundary a third degree polynomial approximation has been used. A complete discussion of this method is given in reference 14. Only a brief outline is given here.

To use the method of the third degree polynomial four grid points near the boundary are used. The stress function is assumed to be of the form

$$\phi = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \tag{17}$$

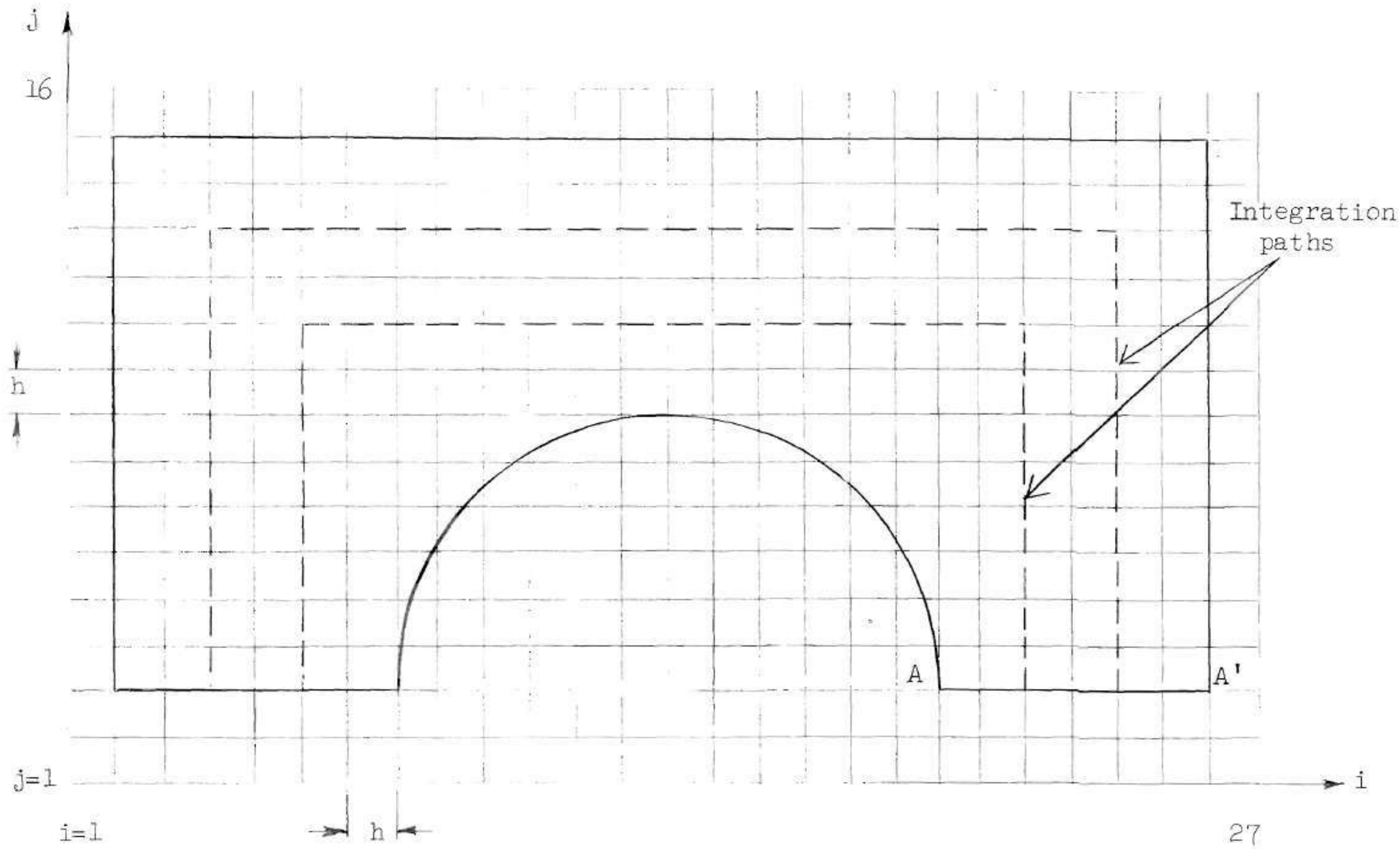
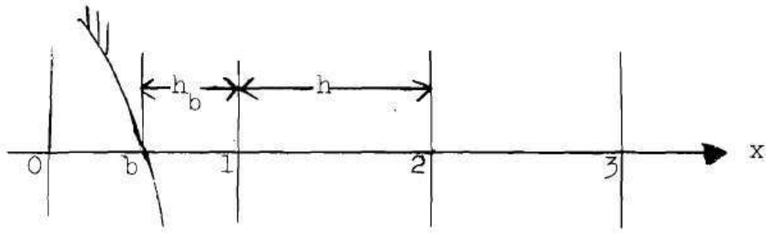


Figure 1. Grid Configuration for Finite Difference Formulation
Showing Paths of Integration.

where x is measured from the boundary. It is easily seen that

$$a_o = \phi_b \quad \text{and} \quad a_1 = \left.\frac{\partial \phi}{\partial x}\right|_b$$



The value of the stress function at the four grid points is given by:

$$\phi_0 = \phi_b + h_b \left.\frac{\partial \phi}{\partial x}\right|_b + a_2(h-h_b)^2 - a_3(h-h_b)^3 \tag{18}$$

$$\phi_1 = \phi_b + h_b \left.\frac{\partial \phi}{\partial x}\right|_b + a_2 h_b^2 + a_3 h_b^3$$

$$\phi_2 = \phi_b + (h_b+h) \left.\frac{\partial \phi}{\partial x}\right|_b + a_2(h_b+h)^2 + a_3(h_b+h)^3$$

$$\phi_3 + \phi_b + (h_b+2h) \left.\frac{\partial \phi}{\partial x}\right|_b + a_2(h_b+2h)^2 + a_3(h_b+2h)^3$$

If the third and fourth equations are solved for a_2 and a_3 and the expressions substituted back into equations one and two, the equations for ϕ_o and ϕ_1 become.

$$\phi_0 = A\phi_b + B\phi_2 - C\phi_3 - Dh \left.\frac{\partial \phi}{\partial x}\right|_b \tag{19}$$

$$\phi_1 = E\phi_b + F\phi_2 - G\phi_3 + Hh \left.\frac{\partial \phi}{\partial x}\right|_b$$

where the constants A, B, C, D, E, F, G, and H are in terms of

h_b/h. For the particular problem all the boundary conditions for the ϕ's on the inside boundary require that

$$\phi_b = \left.\frac{\partial\phi}{\partial x}\right|_b = \left.\frac{\partial\phi}{\partial y}\right|_b = 0$$

Thus the equations for ϕ_0 and ϕ_1 become

$$\phi_0 = B\phi_2 - C\phi_3 \tag{20}$$

$$\phi_1 = F\phi_2 - G\phi_3$$

To use these equations the values for ϕ_2 and ϕ_3 are found by the normal relaxation procedure, and then ϕ_0 and ϕ_1 are calculated from these equations.

Figure 2 shows a section of the curved boundary with the grid superimposed on it. The stress function ϕ is calculated at points such as (19,6) and (18,7) by using the polynomial in both the x and y direction and then using the average value of the ϕ's calculated.

On the outside boundary the boundary conditions are easier to apply However, to apply these conditions use must be made of auxiliary grid points outside the plate.

The method used for solving the biharmonic equations was the Gauss-Seidel relaxation method. For this method the equation is written in the form.

$$\nabla x^4 \phi_{ij} + 2\nabla x^2 \nabla y^2 \phi_{ij} + \nabla y^4 \phi_{ij} = R_{ij} \tag{21}$$

where the R_{ij}'s are called the residuals. The Gauss-Seidel method is started by making an initial guess for the values of ϕ at all the grid points. If the initial guess reduces all the R_{ij}'s to zero the equation

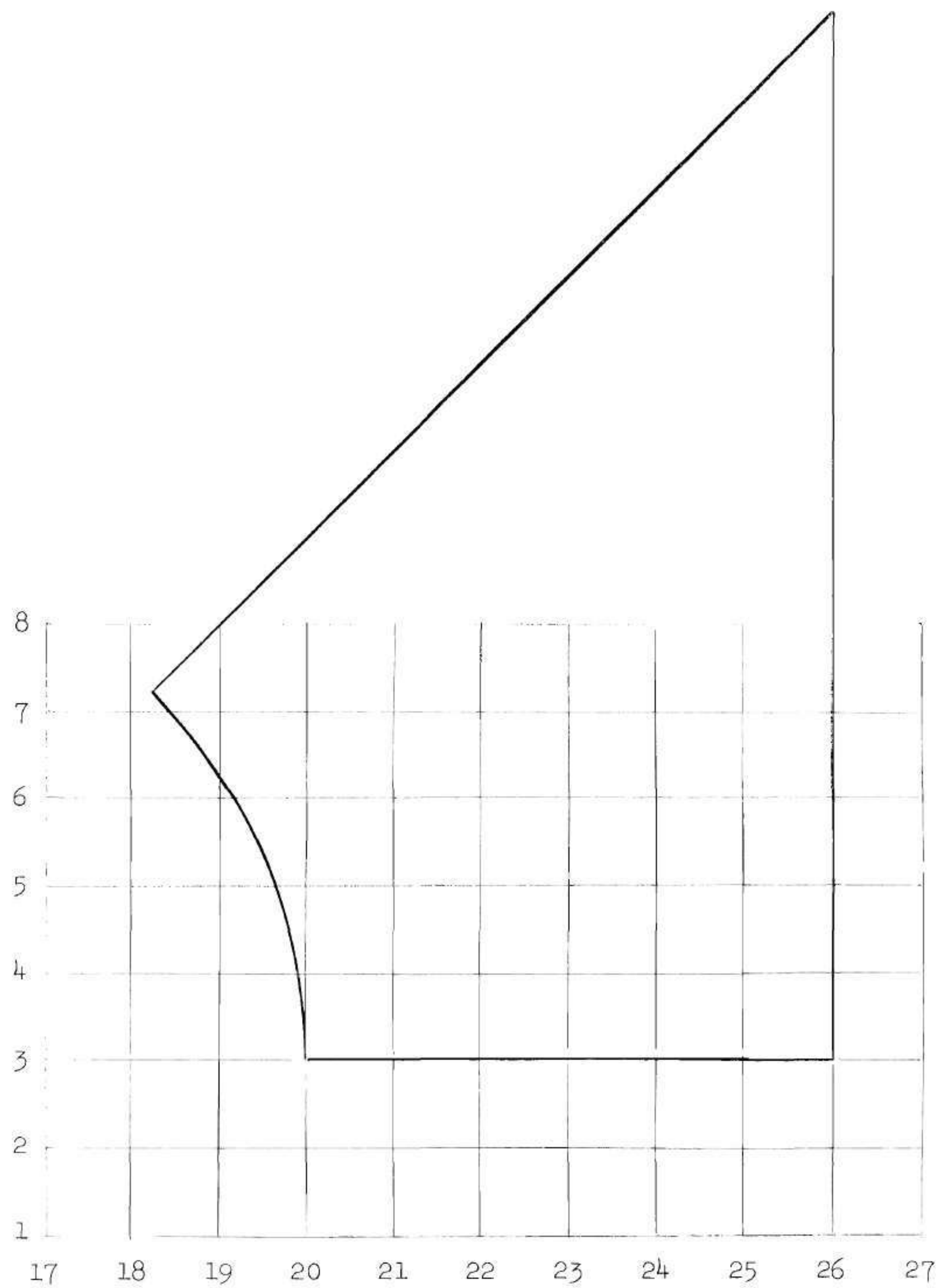


Figure 2. Relation of Grid to Curved Boundary.

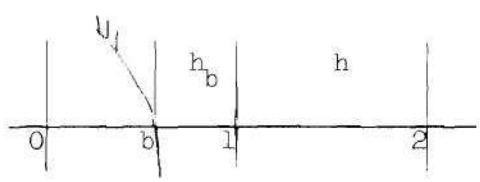
is solved. If the initial guess does not reduce all the R_{ij}'s to zero, then the value of the stress function ϕ_{ij} is adjusted at each point so that R_{ij} is zero at that point. This is done at each point of the grid. This procedure is used over and over until the R_{ij}'s become very small.

The temperature distribution has also been determined numerically. The governing equation here is Laplace's equation. The finite difference form for this equation is

$$\nabla x^2 T + \nabla y^2 T = 0 \tag{21}$$

The boundary conditions are $T = 100$ on the inside boundary and $T = 0$ on the outside boundary.

The approximation for the curved boundary used for this case was much simpler than that used for ϕ since only the value of the function T is specified at the boundary and not the normal derivative.



$$T_o = T_b + \frac{h-h_b}{h_b}(T_b - T_1) \tag{22}$$

The method used for solving the finite difference form of Laplace's equation was also the Gauss-Seidel method.

The evaluation of Michell's equation, equations (9), is now considered. The symmetry of the region, the temperature distribution, and the stress functions makes this task somewhat easier than it may first appear to be.

The determination of the stress functions ϕ_A, ϕ_B, and ϕ_C involves the solution of the homogeneous biharmonic equation. From the boundary conditions for ϕ_A, ϕ_B, and ϕ_C the symmetry, if any, can be determined for each of these stress functions. Using these considerations it can be seen that:

1. ϕ_A is antisymmetrical about the y axis and symmetrical about the x axis.

2. ϕ_B is symmetrical about the y axis and antisymmetrical about the x axis.

3. ϕ_C is symmetrical about both the x and y axes.

Since the Laplacian of a function involves the second derivatives of the function, it possesses the same symmetrical properties as the function.

A similar argument for the temperature distribution shows that it is symmetrical about both the x and y axes.

Now if a path of integration is chosen that is symmetrical about both the x and y axes it can be shown that, when integrating around such a path, the value of the integral will be zero if the integrand is antisymmetric about either the x axis, the y axis, or both. A non-zero value results only when the integrand is symmetric about both the x and y axes.

From these considerations it can be shown that only the following integral from Michell's equations are non-zero.

$$\int_{C_i} \frac{\partial T}{\partial n}\, ds\ , \quad \int_{C_i}\left(y\,\frac{\partial(\nabla^2\phi_A)}{\partial s} + x\,\frac{\partial(\nabla^2\phi_A)}{\partial n}\right) ds$$

$$\int_{C_i}\left(y\,\frac{\partial(\nabla^2\phi_B)}{\partial n} - x\,\frac{\partial(\nabla^2\phi_B)}{\partial s}\right) ds, \quad \int_{C_i} \frac{\partial(\nabla^2\phi_C)}{\partial n}\, ds$$

Substituting these integrals into equations (9) it is seen that the boundary constants a and b are zero. This is not necessarily true for other temperature distributions. Therefore only the value of the integrals

$$\int_{C_i} \frac{\partial T}{\partial n}\, ds \quad \text{and} \quad \int_{C_i} \frac{\partial(\nabla^2 \phi_C)}{\partial n}\, ds$$

are needed to determine boundary constant c. The values of these integrals are equal to twice the values obtained when the integration is carried out over the curve shown in Figure 1.

It is seen also in Figure 1 that the curve is rectangular in nature and always coincides with the grid points. A curve of this type makes the task of evaluating the derivatives $\partial/\partial n$ and $\partial/\partial s$ easier since they may always be taken as x or y derivatives.

The finite difference form for the Laplacian of the stress functions is the same as for the temperature distribution

$$\nabla^2 \phi_{ij} = \frac{\nabla x^2 \phi_{ij} + \nabla y^2 \phi_{ij}}{h^2} \tag{23}$$

For evaluating the derivatives numerically the forms are

$$\frac{\partial \phi_{ij}}{\partial x} = \frac{\phi_{i+1,j} - \phi_{i-1,j}}{2h}, \quad \frac{\partial \phi_{ij}}{\partial y} = \frac{\phi_{i,j+1} - \phi_{i,j-1}}{2h} \tag{24}$$

Since x and y derivatives are used for normal derivatives and since the integrals are line integrals, the sign of the derivatives must be kept consistent with the direction of integration.

The actual integration is carried out using Simpson's Rule

$$\int f(x)dx = \frac{\Delta x}{3} \left\{ f(x_0) + 4f(x_1) + 2f(x_3) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \right\}$$

where n is an even number.

The stress distribution is determined from the solution function ϕ using the finite difference forms of equations (3). These forms are

$$\frac{\partial^2 \phi_{ij}}{\partial x^2} = \frac{\phi_{i+1,j} - 2\phi_{ij} + \phi_{i-1,j}}{h^2} \tag{25}$$

$$\frac{\partial^2 \phi_{ij}}{\partial y^2} = \frac{\phi_{i,j+1} - 2\phi_{ij} + \phi_{i,j-1}}{h^2}$$

$$\frac{\partial^2}{\partial x \partial y} = \frac{\phi_{i+1,j+1} + \phi_{i-1,j-1} - \phi_{i+1,j-1} - \phi_{i-1,j+1}}{4h^2}$$

The entire numerical procedure was programmed for and carried out on the Burrough's 220 digital computer. The programs used to carry out this numerical procedure are given in Appendix C.

Results

The results of the numerical procedure are given in Figures 16, 17, 18, 19, and 20. Figures 16, 17, and 18 give the values of the stress functions ϕ_A, ϕ_B, and ϕ_C. Figures 19 and 20 give the values of the σ_x and σ_y stresses respectively.

Figure 3 is a plot of σ_x and σ_y versus the distance across section AA'. The two points which give the σ_y stress at the inside boundary tend to scatter due to the difficulty in approximating the boundary conditions on the curved boundary. Since the second point is away from the boundary it is reasonable to suspect that this point is the more accurate of the two. By drawing the best average curve to fit the data points, the stresses at the inside and outside of the plate at section AA'

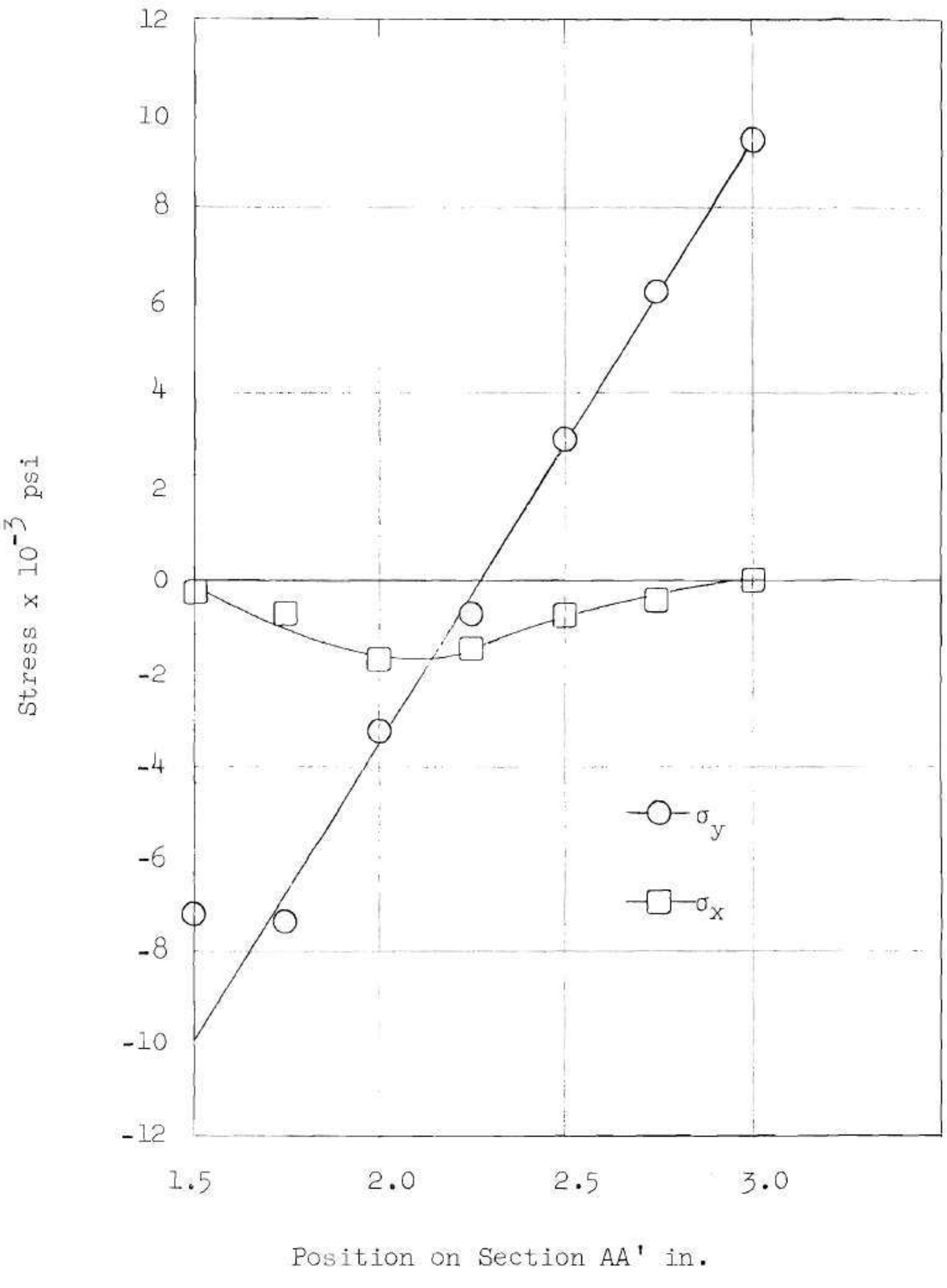


Figure 3. Stress vs. Distance Along Section AA'.

$$\sigma_{yi} = -\ 10{,}000 \text{ psi}$$

$$\sigma_{yo} = \quad 9{,}500 \text{ psi}$$

The point at which the σ_x and σ_y curves cross is the point where the maximum shear stress is zero. A graph of twice the maximum shear stress versus the distance across section AA' is given in Figure 4.

There is no check on the accuracy of this numerical procedure against a closed form analytical solution for this problem. However, in Chapter V there is a comparison of these stresses against stresses obtained experimentally.

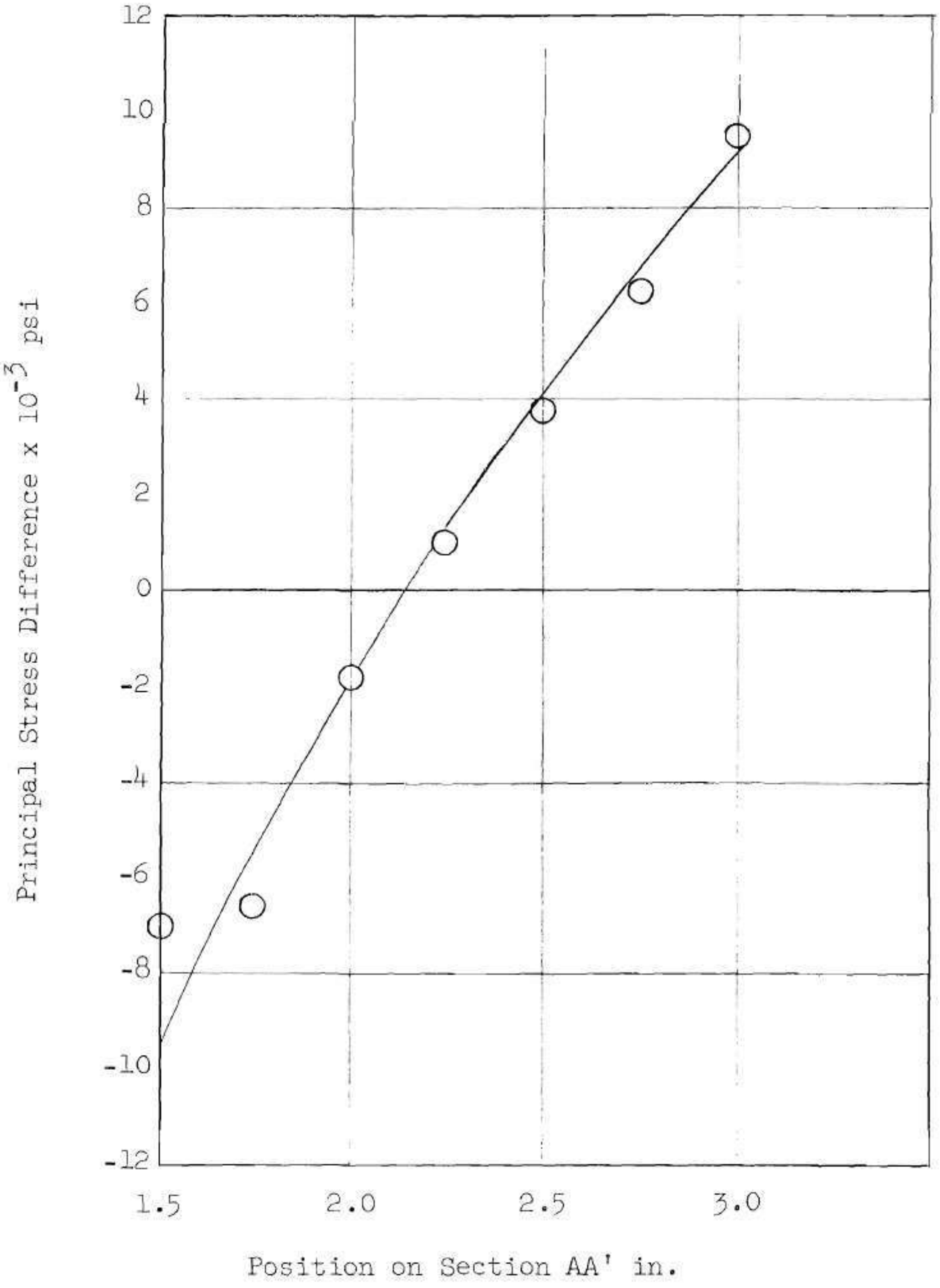


Figure 4. Principal Stress Difference vs. Distance along Section AA'.

CHAPTER IV

THERMAL STRESSES BY PHOTOELASTICITY

There are several ways of determining thermal stresses photoelastically. In this study the method which employs the dislocation analogy has been used. This method may be used to determine the thermal stresses in a multiply connected body that has a steady state temperature distribution. Stresses equal and opposite to the thermal stresses are mechanically induced in a model by cutting a slit in the model to form a simply connected region and then displacing the cut edges relative to each other. The amount of the displacements are determined from the particular temperature distribution that is being considered. The dislocation analogy was introduced by Biot[5] and was first used by Weibel.[6] The results obtained by Biot which make up the dislocation analogy are given in the following summary.

1. There are no stresses in an elastic solid in which a steady state two-dimensional heat flow condition exists, except for a multiply connected body such as a flat plate with a hole in it.

2. For the case of a simply connected body, if the longitudinal stresses are zero (plane stress), the thermal expansion ϵ is given by

$$\epsilon = \alpha T \tag{26}$$

The quantity ϵ and the corresponding rotation ω satisfy the following equations

$$\frac{\partial \epsilon}{\partial x} = \frac{\partial \omega}{\partial y} \quad , \quad \frac{\partial \epsilon}{\partial y} = - \frac{\partial \omega}{\partial x} \tag{27}$$

For a derivation of these equations see Appendix B.

3. The change in rotation between two points P_1 and P_2 is given by

$$\omega_2 - \omega_1 = \int_1^2 d\omega = \int_1^2 \frac{\partial \omega}{\partial x} dx + \frac{\partial \omega}{\partial y} dy$$

$$= \int_1^2 - \frac{\partial \epsilon}{\partial y} dx + \frac{\partial \epsilon}{\partial x} dy$$

$$= \alpha \int_1^2 \frac{\partial T}{\partial y} (-dx) + \frac{\partial T}{\partial y} dy = \alpha \int \frac{\partial T}{\partial n} ds$$

where n is the direction normal to the path of integration. Hence the change in rotation from point 1 to point 2 is proportional to the heat flow.

4. The stressed multiply connected body may be changed to an unstressed simply connected body by a cut between the inner and outer boundaries. Displacements u and v of a point at the free edge of the cut are given by the equations

$$u = \int_C du = \int_C \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$v = \int_C dv = \int_C \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

where C is a closed curve that encloses the inner boundary. See Figure 5.

5. Displacements u and v, and the rotation ω, when calculated from a knowledge of the temperature distribution may be applied to

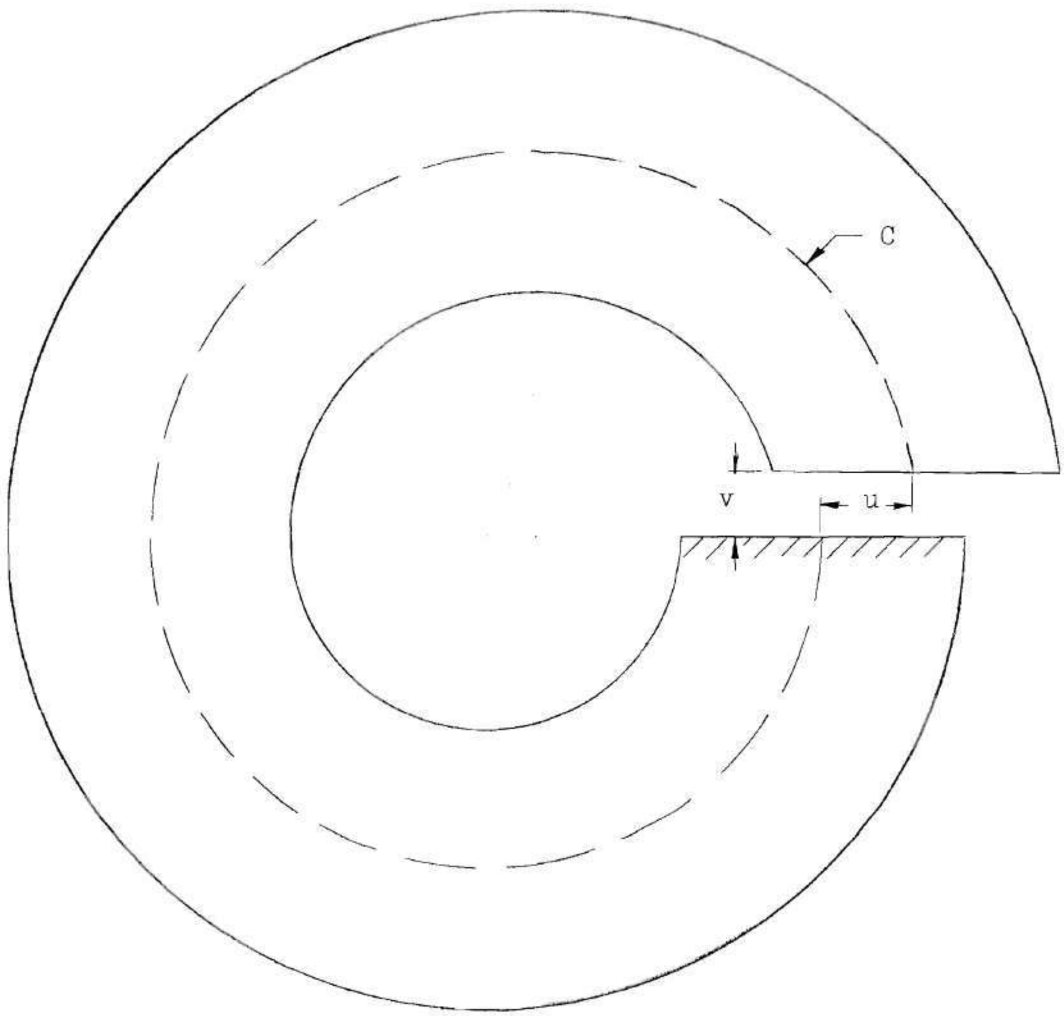


Figure 5. u and v Displacements at the Free Edge of a Slit.

a photoelastic model giving stresses which are proportional to, but opposite in sign from the thermal stresses in the plate.

Evaluation of the Displacements u and v, and the Rotation ω

Determination of u, v, and ω consists of carrying out the integration in the equations

$$\omega = \alpha \int_C \frac{\partial T}{\partial n} \, ds \tag{28}$$

$$u = \int_C \frac{\partial u}{\partial x} \, dx + \frac{\partial u}{\partial y} \, dy$$

$$v = \int_C \frac{\partial v}{\partial x} \, dx + \frac{\partial v}{\partial y} \, dy$$

In order to get the integrands in a more convenient form some rearranging is necessary.

Using the equilibrium equations for internal stress, equations (1), the compatibility equation for strain, equation (5), the stress-strain relations, equations (2), and letting

$$\sigma_x = \sigma_y = \tau_{xy} = 0$$

in a simply connected region, the following relations can be obtained.

$$\frac{\partial u}{\partial x} = \epsilon \ , \qquad \frac{\partial v}{\partial y} = \epsilon$$

$$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0$$

By definition

$$\omega = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Combining the last two equations gives

$$\frac{\partial u}{\partial y} = - \omega , \quad \frac{\partial v}{\partial x} = \omega$$

Then

$$u = \int_C \epsilon dx - \omega dy \tag{29}$$

$$v = \int \omega dx + \epsilon dy$$

or

$$du = \epsilon dx - \omega dy$$

$$dv = \omega dx + \epsilon dy$$

$$d\omega = \alpha \frac{\partial T}{\partial n} ds$$

or the above equations in finite difference form become

$$\Delta u = \epsilon \Delta x - \omega \Delta y \tag{30}$$

$$\Delta v = \omega \Delta x - \epsilon \Delta y$$

$$\Delta \omega = \alpha \frac{\Delta T}{\Delta n} \Delta s$$

Consider $\Delta\omega$ which is proportional to the heat flow Δq. Any method which can be used to solve for the heat flow in the plate can also be used to solve for the rotation. The method of curvilinear squares may be used to give a network of constant temperature lines and heat flow lines.

Figure 6 shows a plane body with the curvilinear squares drawn. Now using the fact that the rotation ω is proportional to the heat flow it can be shown that

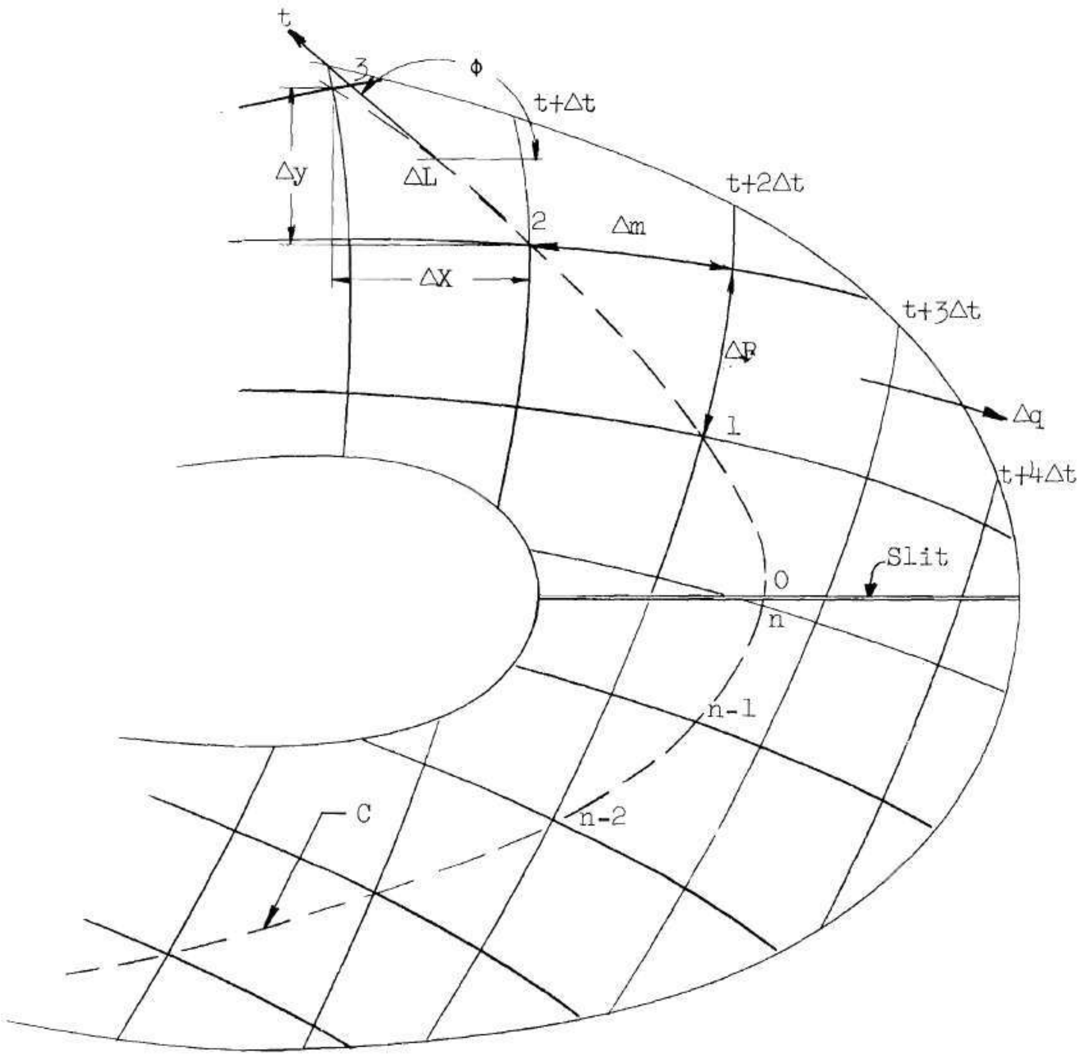


Figure 6. Plane Body with Curvilinear Squares.

$$\Delta\omega = \alpha \frac{\Delta t}{\Delta m} \Delta P$$

for $\frac{\Delta P}{\Delta m} = 1$

$$\Delta\omega = \alpha \, \Delta t$$

and for a curve C that surrounds the inner boundary the rotation at the free edge of the slit is given by

$$\omega = \sum_{i=1}^{n} \Delta\omega_i \tag{31}$$

$\Delta\omega_i$ represents the change in rotation from where C crosses one heat flow line to where C crosses the next heat flow line.

For the u and v displacements

$$\Delta u_i = \epsilon_i \Delta x_i - \omega_i \Delta y_i \tag{32}$$

$$\Delta v_i = \omega_i \Delta x_i + \epsilon_i \Delta y_i$$

From the first equation consider the term $\omega_i \Delta y_i$

$$\sum_{1}^{n} \omega_i \Delta y_i = \omega_1(y_1 - y_o) + \omega_2(y_2 - y_1) + \omega_3(y_2 - y_2) + \ldots + \omega_n(y_n - y_{n-1})$$

Now

$$\omega_2 = \omega_1 + \Delta\omega, \; \omega_3 = \omega_2 + \Delta\omega, \; \ldots, \; \omega_n = \omega_{n-1} + \Delta\omega$$

If the Δt's between the isotherms are constant for all of the isotherms then the $\Delta\omega$'s are constant since

$$\Delta\omega \approx \Delta\omega$$

Then

$$\omega_1 = \Delta\omega, \ \omega_2 = 2\Delta\omega, \ \omega_3 + 3\Delta\omega, \ \ldots, \ \omega_n = n\Delta\omega$$

and

$$\sum_1^n \omega_1 \Delta y_i = \Delta\omega(y_1 - y_0) + 2\Delta\omega(y_2 - y_1) + \ldots + n\Delta\omega(y_n - y_{n-1})$$

$$= -\Delta\omega y_1 \quad \Delta\omega y_2 - \ldots - \Delta\omega y_{n-1} - \Delta\omega y_0 + n\Delta\omega y_n$$

But

$$y_n = y_0 = 0$$

so that

$$\sum_1^n \omega_i \Delta y_i = - \sum_1^n y_i \Delta\omega \tag{33}$$

Now consider the term $\epsilon_i \Delta x_i$

$$\epsilon_i = \alpha T_i \ , \quad \Delta x_i = \Delta L_i \cos \phi_i$$

The equation for the u displacements becomes

$$\Delta u_i = \Delta L_i \alpha T_i \cos \phi_i + \Delta\omega y_i \tag{34}$$

similarly

$$\Delta v_i = \Delta L_i \alpha T_i \sin \phi_i - \Delta\omega x_i \tag{35}$$

It should be kept in mind when using these final equations that $\Delta\omega$ was set up to be a constant and that the curve C should always cross two different heat flow lines when passing through a curvilinear square. It is usually more convenient to choose an isotherm for the curve C when this is possible.

Then the final displacements which are to be imposed on the model are given by

$$\omega = \sum_{1}^{n} \Delta\omega_i = n\Delta\omega \tag{36}$$

$$u = \sum_{1}^{n} (\Delta L_i \alpha T_i \cos \phi_i + \Delta\omega y_i)$$

$$v = \sum_{1}^{n} (\Delta L_i \alpha T_i \sin \phi_i - \Delta\omega x_i)$$

Displacements for the Particular Problem

For the particular problem of the square plate with a center circular hole and with a steady state temperature distribution of $T_i = 100$ degrees F and $T_o = 0$ degrees F, the set of curvilinear squares is given in Figure 7.

From the symmetry of the temperature distribution and from the choice of orientation of the x and y axes, the u displacement is zero. Likewise the first term of the equation for the v displacement vanishes. Then for the particular problem the equations for calculating the displacements are

$$\omega = \sum_{1}^{n} \Delta\omega_i = n\Delta\omega$$

$$u = 0$$

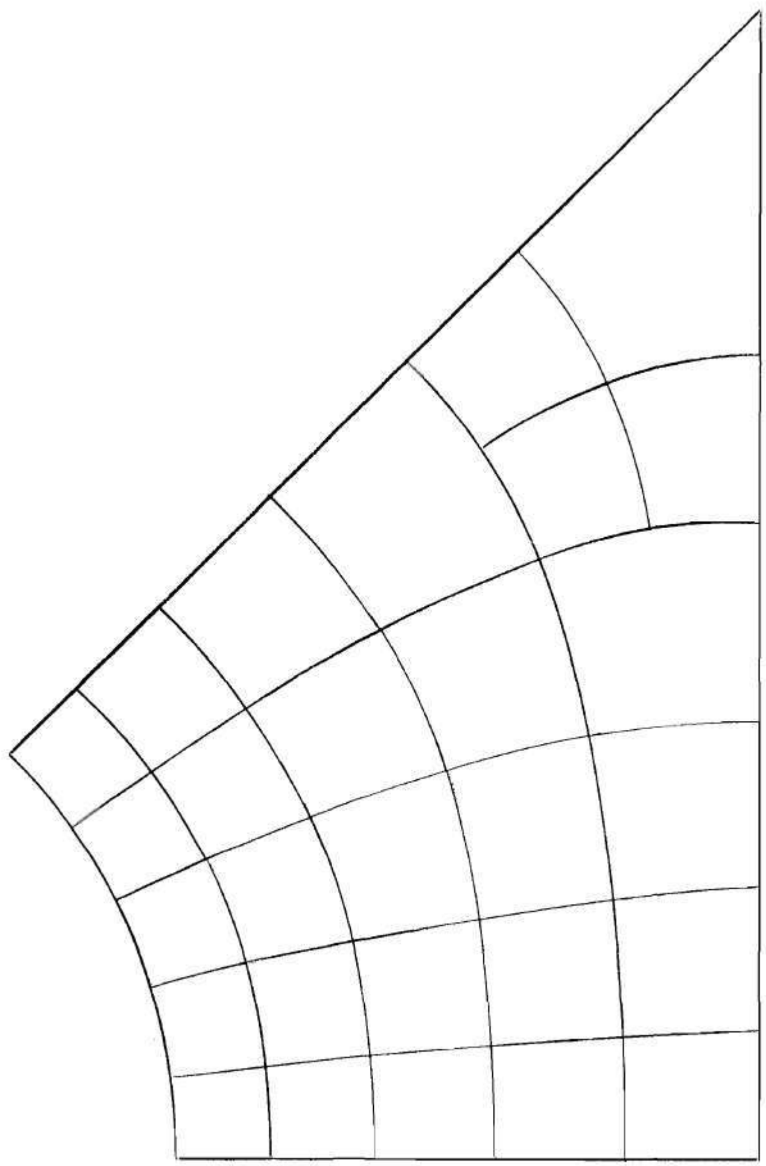


Figure 7. The Curvilinear Squares for the Particular Problem.

$$v = \sum_{1}^{n} (-\Delta\omega x_i)$$

The curves used for calculating the displacements were the 100, 80, 60, 40, 20, and 0 degree isotherms. The values calculated were

$$v_{100} = 7.83 \times 10^{-3} \text{ in}$$

$$v_{80} = 8.93 \times 10^{-3} \text{ in}$$

$$v_{60} = 10.15 \times 10^{-3} \text{ in}$$

$$v_{40} = 11.87 \times 10^{-3} \text{ in}$$

$$v_{20} = 13.70 \times 10^{-3} \text{ in}$$

$$v_{0} = 15.60 \times 10^{-3} \text{ in}$$

The slit was cut in the model as shown in Figure 8. A plot of the v displacement versus position along the slit is given in Figure 9. The slope of the curve drawn through the points is calculated to be 5.17×10^{-3}. This is the numerical value for $\partial v/\partial x$ along the slit. An independent calculation of a value for ω gives a value of 5.20×10^{-3}. Since $\omega = \frac{\partial v}{\partial x}$ there is good agreement between the values.

The Experimental Procedure

The photoelastic plastic used for the experimental investigation was CR-39. The selection of this material was based on photoelastic properties and availability.

The photoelastic model had dimensions one and one-third times as large as the dimensions of the particular problem. The purpose of using

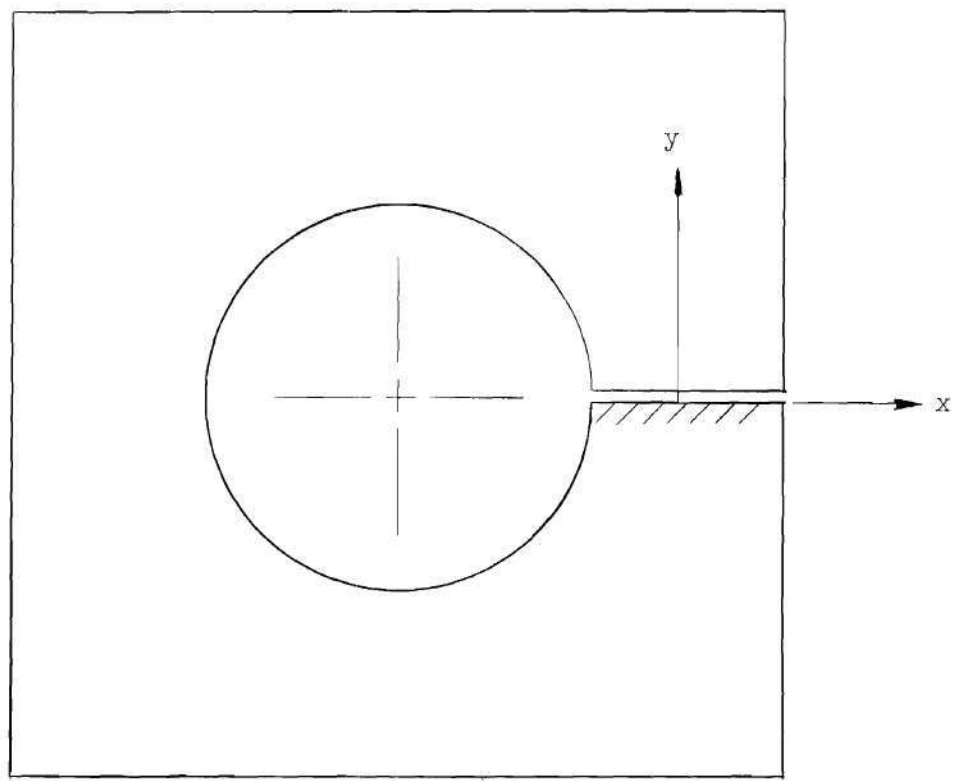


Figure 8. Location of the Slit for the Particular Problem.

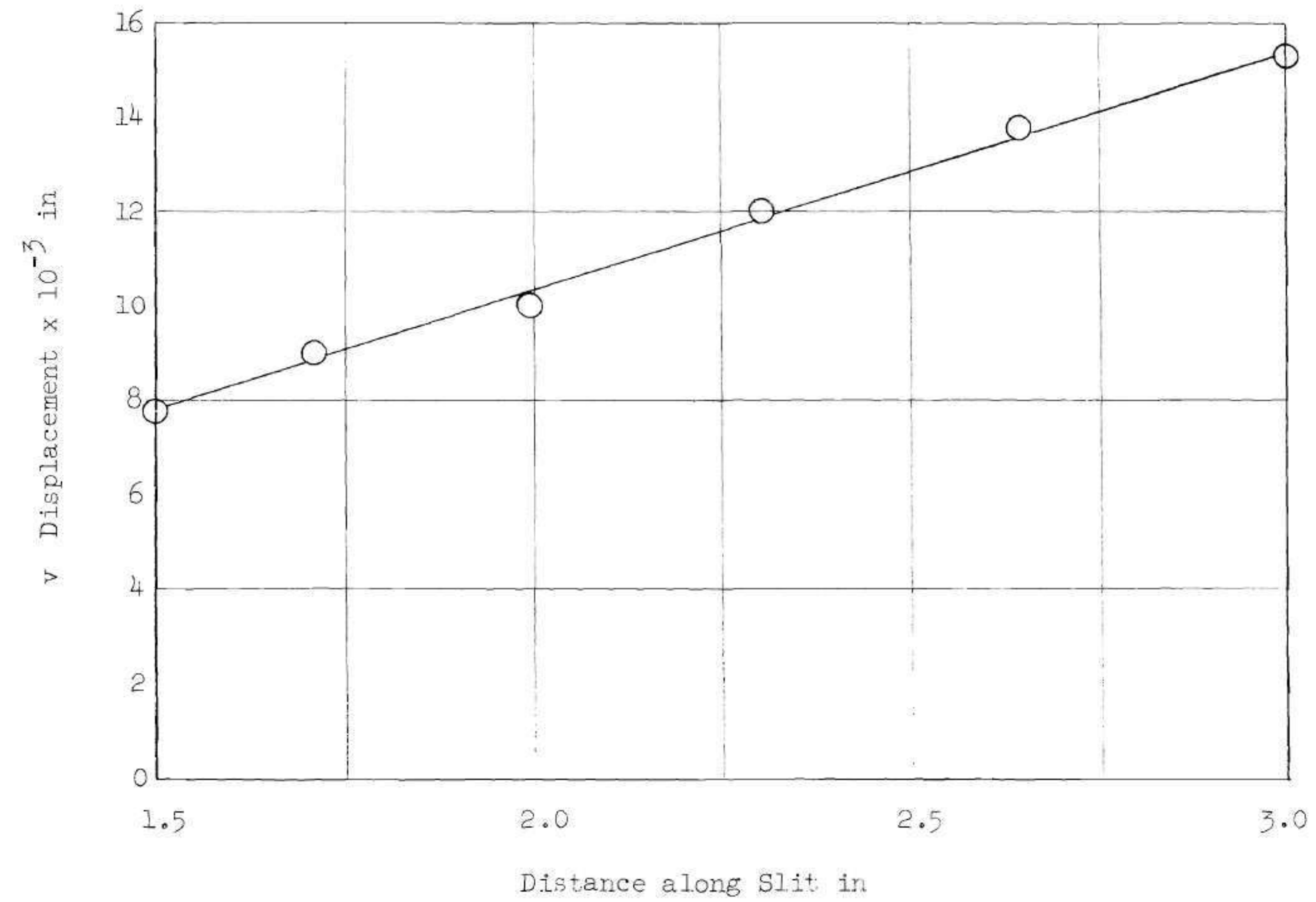


Figure 9. v vs. Distance along the Slit.

a large model was to decrease the error. Models larger than the model used would have been too cumbersome to manage.

The calculated displacements were applied to the model by means of a photoelastic deformeter. Using the deformeter the displacements could be applied with very small error. The deformeter, shown in Figure 10 was similar to the one used by Weibel. Loading was accomplished by driving taper pins into the 90 degrees V grooves. The deformeter can easily be adapted to give u displacements also, but this was not necessary for this particular problem.

If the calculated displacements given on page 35 were applied to the model the stresses induced would probably not give a useable stress pattern. For most cases a factor K must be used to give displacements that will induce the best stress pattern. This factor K must be determined by trial and error. Also if the model is not the same size as the prototype, this must be taken into account by a factor S. If the linear thermal coefficient of expansion α of the model has been used to calculate the displacements, then this must be taken into account when determining the actual stresses in the prototype. Then the actual stresses in the prototype are given by

$$(\sigma_1 - \sigma_2) = \frac{nf}{d}\left(\frac{EP}{Em} \times \frac{\alpha P}{\alpha m} \times \frac{1}{K} \times \frac{1}{S}\right)$$

where n is the fringe order, f is the fringe constant (lb/in-order), and d is the thickness of the material.

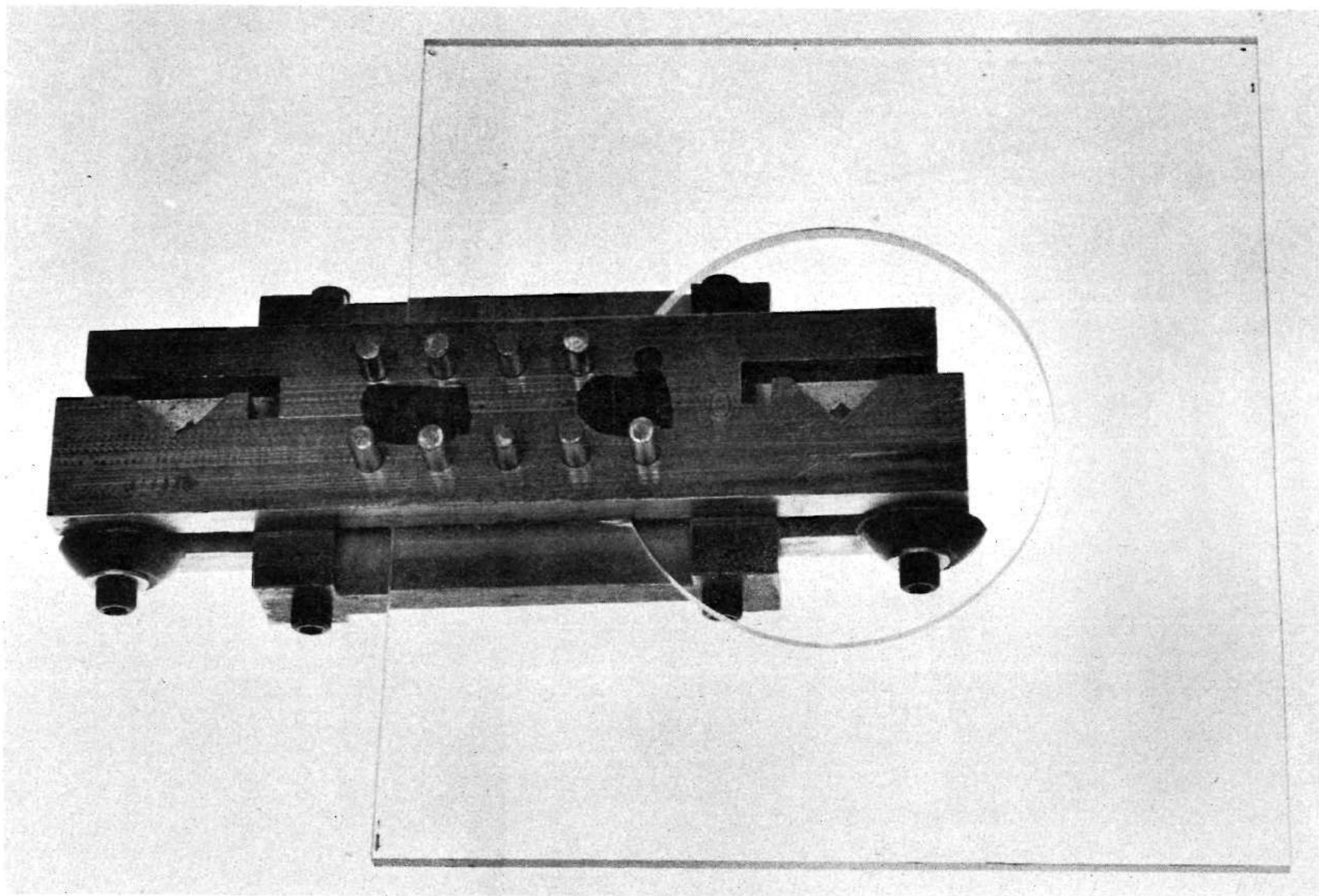


Figure 10. Photoelastic Deformeter.

Results

A typical stress pattern is given in Figure 11. A plot of the fringe order versus the distance across section AA' is given in Figure 12. It is seen in this figure that the experimental data points have a slight scatter, but that there is no difficulty in fitting a curve to the points. This scatter is probably due to the fact that the isochromatic lines, which give the stress, appear as broad black lines on the photograph. This makes the exact determination of the lines' relative positions difficult.

Although the fringe orders determine the magnitude of the difference in the principal stresses, they may be used to give the stresses at the inside and outside boundary at section AA' directly for this particular problem. These stresses are calculated to be

$$\sigma_i = 10{,}800 \text{ psi}$$

$$\sigma_o = 9{,}000 \text{ psi}$$

The shear difference method,[11] which is based on the equations

$$\sigma_x = (\sigma_x)_o - \int \frac{\partial \tau_{xy}}{\partial y}\, dx \qquad (37)$$

$$\sigma_y = (\sigma_y)_o - \int \frac{\partial \tau_{xy}}{\partial x}\, dy$$

where $(\sigma_x)_o$ and $(\sigma_y)_o$ represent the values of σ_x and σ_y at a given location, may be used to calculate the stresses at any point across section AA'. However, looking at the isoclinics, Figure 13, it is seen that the isotropic point in section AA' will make this calculation difficult. The use of equations (37) depends on the calculation of the shear stress τ_{xy} across some section close to and parallel to section AA'. The shear stress is given by

$$\tau_{xy} = \frac{(\sigma_1 - \sigma_2)}{2} \sin 2\theta$$

where σ_1 and σ_2 are the principal stresses and θ is the acute angle between the direction of σ_1 and the plane on which τ_{xy} acts. Across a section close to section AA' the quantity $\sin 2\theta$ is very small when $(\sigma_1 - \sigma_2)$ is large and the quantity $(\sigma_1 - \sigma_2)$ is very small when $\sin 2\theta$ is large. This causes one to suspect that the σ_x stresses given by

$$\sigma_x = (\sigma_x)_o - \int \frac{\partial \tau_{xy}}{\partial y} \, dx$$

would be small also, since τ_{xy} is small at a section close to section AA' and $\tau_{xy} = 0$ along section AA'. If this is true, then approximate values for σ_y can be obtained directly from the stress pattern.

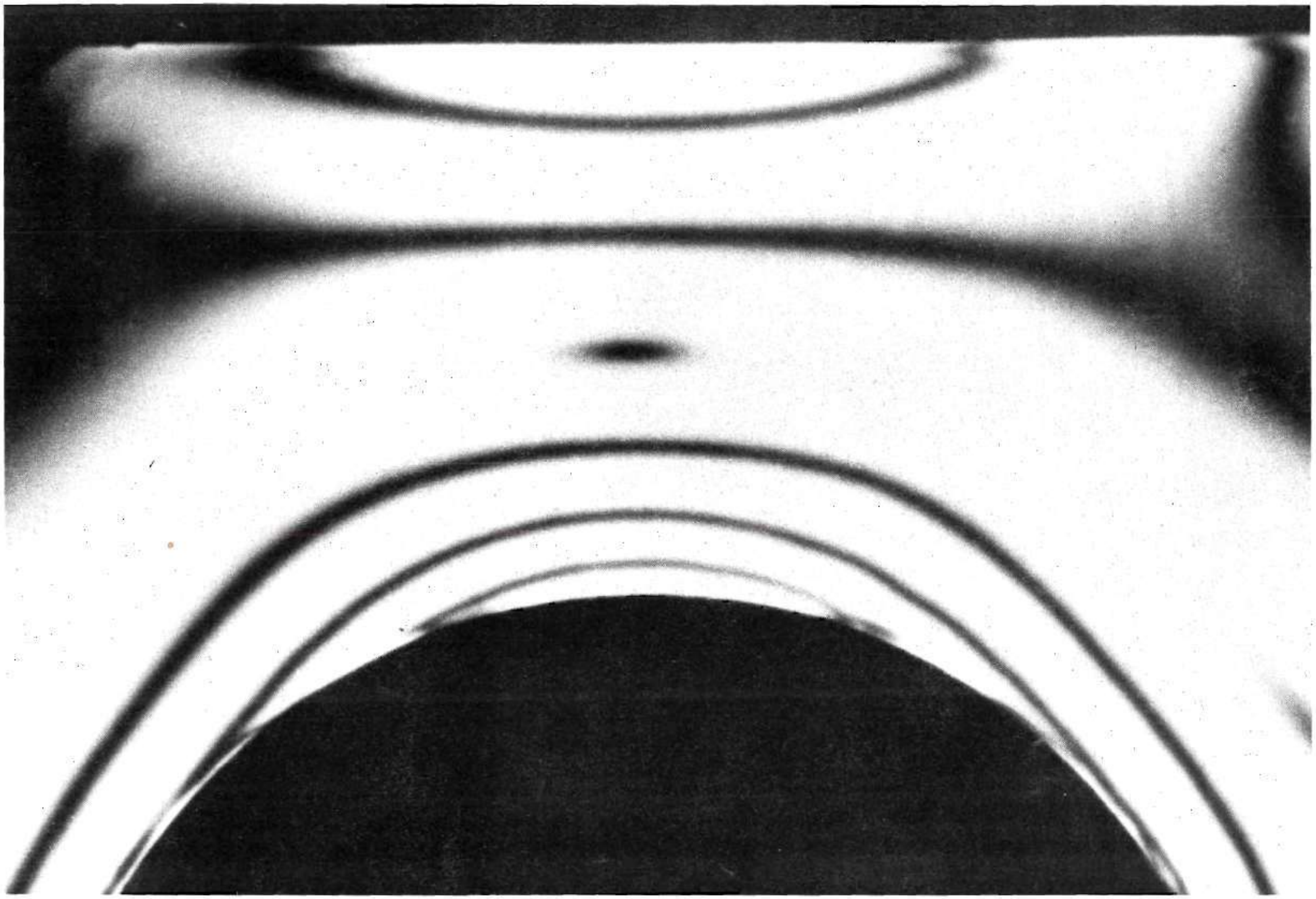


Figure-11 Typical Stress Pattern

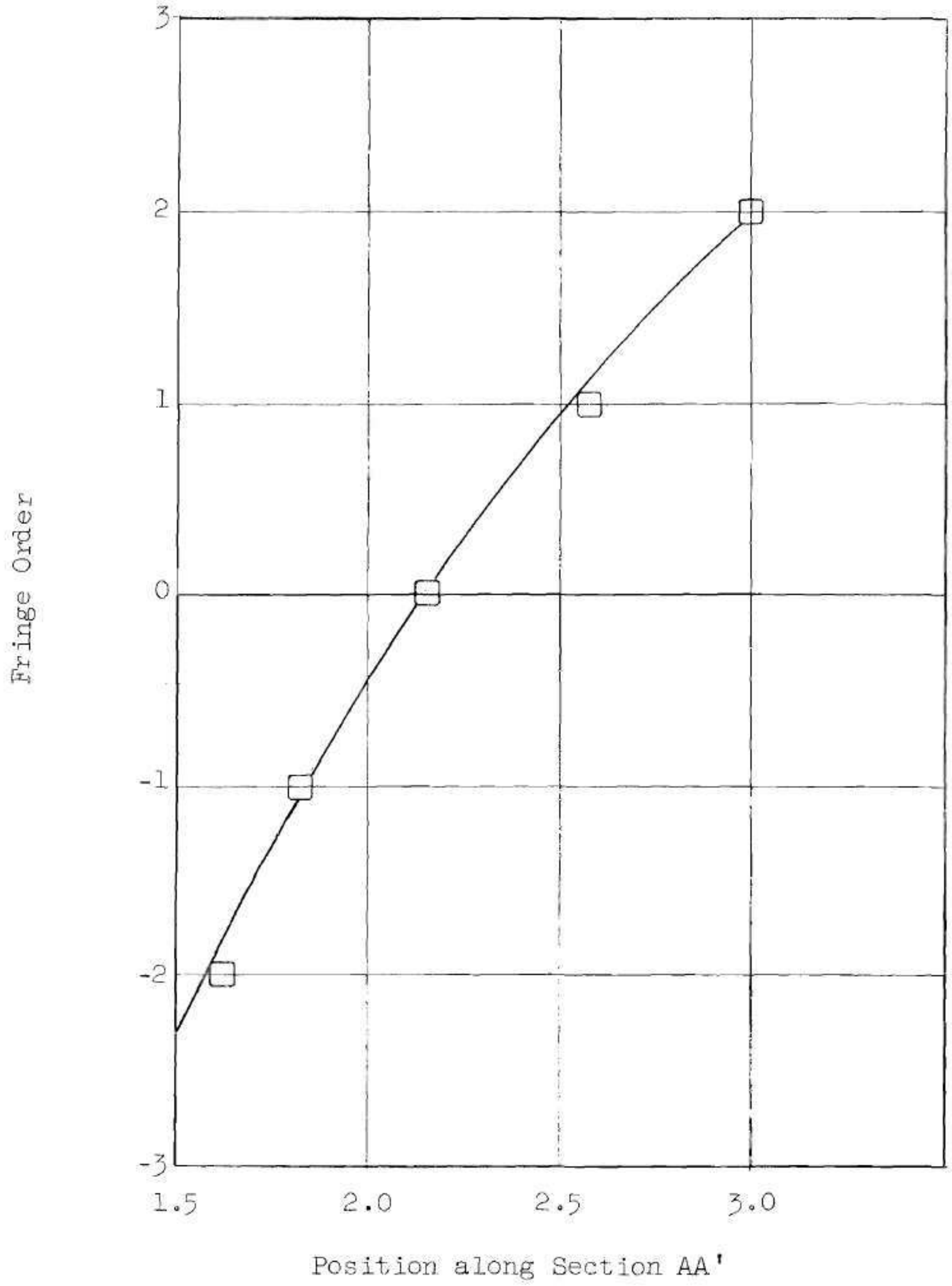


Figure 12. Fringe Order vs. Position along Section AA'.

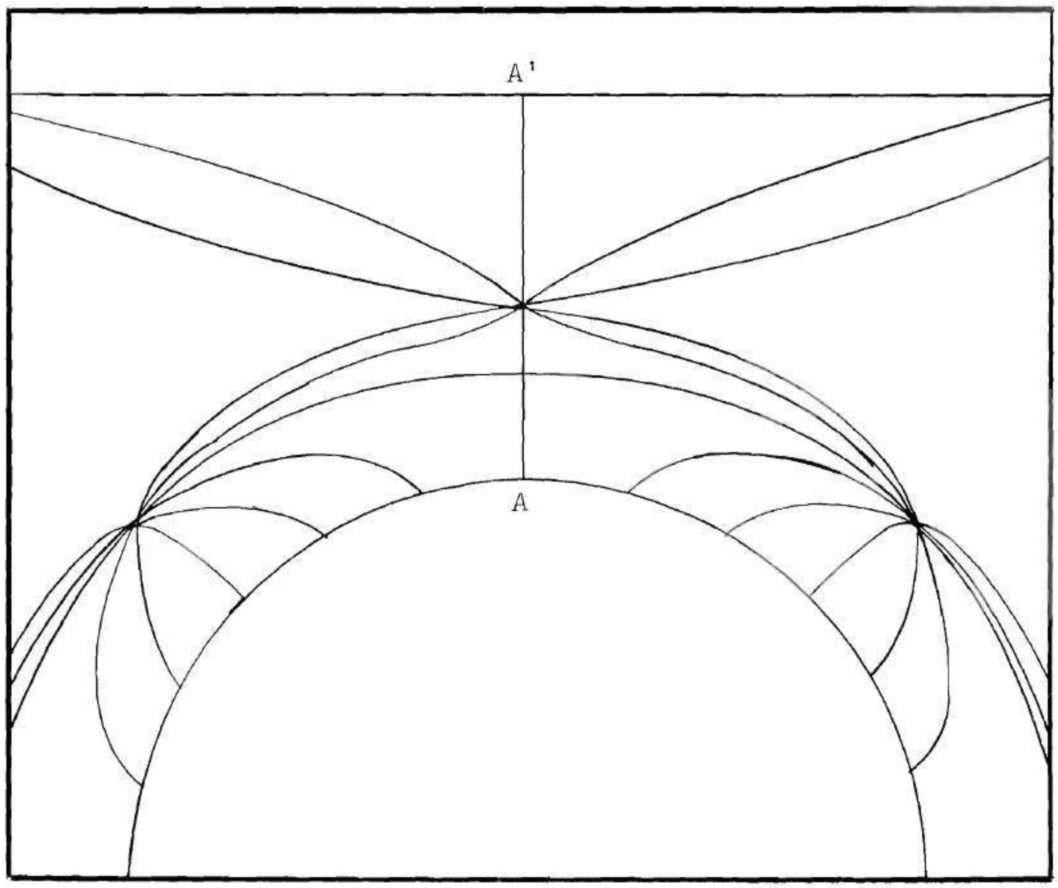


Figure 13. Isoclinics.

CHAPTER V

DISCUSSION

Conclusions

Figure 14 shows a graph of the principal stress difference versus distance across section AA' by the two different methods. As is seen in the figure there is a close correlation of the stresses determined by the two methods. Although no closed form analytical solution has been available to use for a comparison, the close correlation between the two different methods tends to establish that the stresses calculated are reasonably good approximations to the actual stresses.

It is believed that both of these methods for calculating the thermal stresses in multiply connected bodies are practical and will yield reliable results. Both of the methods have advantages and disadvantages.

The numerical methods offers the advantage that it gives the stresses, σ_x, σ_y and τ_{xy} at many points in the body directly and no additional computations are needed. The entire problem can be carried out with one computer program if the computer used has sufficient memory space. Another advantage of this method is that it can be used to solve for thermal stresses even when the temperature distribution is not steady state.

The major disadvantage of this method is the computer time necessary to carry out the solution. The computer time for this problem is about 2 hours and this will increase as more general shapes are used. The computer time will also increase with refinement of the grid size. Most of the computer time is used in solving the biharmonic equation. If some method could be found for solving the biharmonic equation in a multiply connected region more quickly, then the usefulness of this method would be greatly increased.

The photoelastic method offers a relatively easy way of calculating the thermal stresses in a multiply connected region with steady state heat transfer if the equipment is available. As pointed out in Chapter IV this method cannot be used if the temperature distribution does not satisfy Laplace's equation. The primary disadvantage of this method is the difficulty in calculating the stresses at particular points in the body. Although there are several methods for doing this most of them are time-consuming and reduce the accuracy.

It appears that the choice of method would depend on the application since the strong points of one method are the weak points of the other and vice versa. If a stress distribution over an entire region is needed, then the numerical methods appear best, however, if only the stress at several isolated points is needed, the photoelastic method is the easiest and quickest.

Recommendations

The numerical method should be investigated further by:

1. Changing the grid size to see what this effect will have on the stresses.

2. Employing different approximations for the boundary conditions on the curved boundary to determine if a simpler approximation will give satisfactory results.

3. Investigating and/or refining other methods of solving the biharmonic equation in order to reduce computer time.

Although the photoelastic methods are well established it is believed that the solution of more problems by this method will provide additional understanding that will help to iron out some of the difficulties.

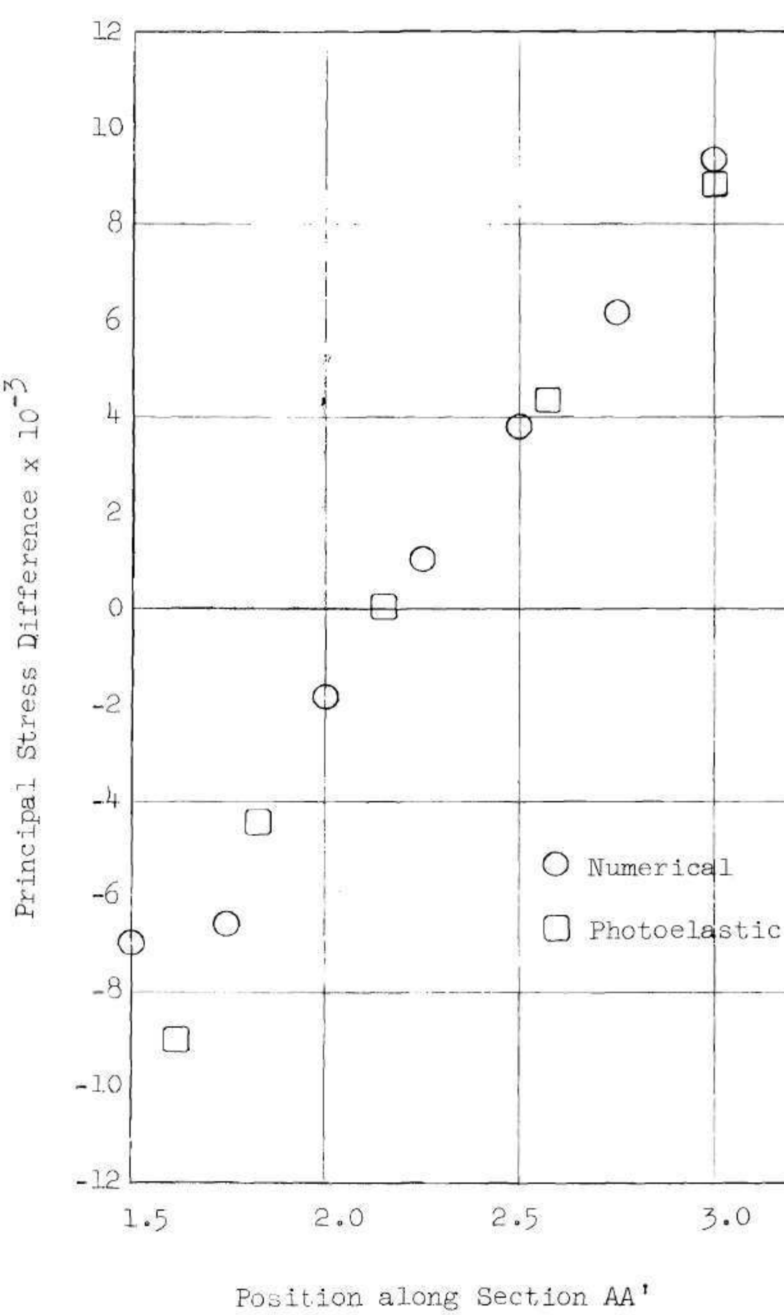


Figure 14. Principal Stress Difference vs. Position along Section AA'.

APPENDIX A

ROLE OF THE COMPATIBILITY EQUATION (5) IN DETERMINING THE SINGLE-VALUEDNESS OF THE DISPLACEMENTS u AND v

Consider the line integrals

$$\int_{C_r} du, \quad \int_{C_r} dv$$

where C_r is any closed curve in a free slice

$$\int_{C_r} du = \int_{C_r} \left(\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \right)$$

Now

$$\epsilon_x = \frac{\partial u}{\partial x}$$

and from the equations

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

$$2\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

we get

$$\frac{\partial u}{\partial y} = \frac{1}{2} \gamma_{xy} - \omega$$

$$\frac{\partial v}{\partial x} = \frac{1}{2} \gamma_{xy} + \omega$$

Then

$$\int_{C_r} du = \int_{C_r} (\epsilon x dx + \frac{1}{2}\gamma_{xy} dy) - \int_{C_r} \omega dy$$

Now

$$\int_{C_r} \omega dy = y_r \int_{C_r} d\omega - \int_{C_r} y d\omega$$

where y_r is the y coordinate of the starting point of integration on C_r.

$$y_r \int d\omega = y_r \int_{C_r} \left(\frac{\partial\omega}{\partial x} dx + \frac{\partial\omega}{\partial y} dy \right)$$

$$= y_r \int \left[\left(\frac{1}{2} \frac{\partial\gamma_{xy}}{\partial x} - \frac{\partial^2 u}{\partial x \partial y} \right) dx + \left(\frac{\partial^2 v}{\partial x \partial y} - \frac{1}{2} \frac{\partial\gamma_{xy}}{\partial y} \right) dy \right]$$

$$= y_r \int \left[\left(\frac{1}{2} \frac{\partial\gamma_{xy}}{\partial x} - \frac{\partial\epsilon_x}{\partial y} \right) dx + \left(\frac{\partial\epsilon_y}{\partial x} - \frac{1}{2} \frac{\partial\gamma_{xy}}{\partial y} \right) dy \right]$$

Similarly

$$\int_{C_r} y d\omega = \int_{C_r} \left[y \left(\frac{1}{2} \frac{\partial\gamma_{xy}}{\partial x} - \frac{\partial\epsilon_x}{\partial y} \right) dx + y \left(\frac{\partial\epsilon_y}{\partial x} - \frac{1}{2} \frac{\partial\gamma_{xy}}{\partial y} \right) dy \right]$$

Then

$$\int_{C_r} du = \int_{C_r} (\epsilon x dx + \frac{1}{2}\gamma_{xy} dy) + \int_{C_r} y d\omega - y_r \int_{C_r} d\omega$$

$$= \int_{C_r} \left[\epsilon_x + (y - y_r) \left(\frac{1}{2} \frac{\partial\gamma_{xy}}{\partial x} - \frac{\partial\epsilon_x}{\partial y} \right) \right] dx$$

$$+ \int_{C_r} \left[\frac{1}{2} \gamma_{xy} + (y - y_r) \left(\frac{\partial\epsilon_x}{\partial x} - \frac{1}{2} \frac{\partial\gamma_{xy}}{\partial y} \right) \right] dy$$

Applying Green's Theorem to this equation gives

$$\int_{C_r} du = \iint_{R_r} \left[(y-y_r) \left(\frac{\partial^2 \epsilon_y}{\partial x^2} + \frac{\partial^2 \epsilon_x}{\partial y^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \right) \right] dxdy - \sum \int_{C_i} du$$

where R_r is the region enclosed by C_r and the summation is over all internal boundaries C_i enclosed by C_r.

If the region enclosed by C_r is simply connected the summation

$$\sum \int_{C_i} du$$

does not exist. Then if

$$\frac{\partial^2 \epsilon_y}{\partial x^2} + \frac{\partial^2 \epsilon_x}{\partial y^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

The integral

$$\int_{C_r} du = 0$$

and the displacements u is single-values. Then for a simply connected region the compatibility equation provides a necessary and sufficient condition for the u displacement to be single-valued. For a multiply connected region the summation of the integrals around the internal boundaries must also be equal to zero to guarantee that the u displacement is single-valued. Then the compatibility equation provides a necessary but not a sufficient condition for the u displacement to be single-valued in a multiply connected region.

A similar result can be obtained for the v displacement and the rotation ω.

APPENDIX B

DERIVATION TO THE EQUATIONS RELATING THE ROTATION ω TO THE THERMAL EXPANSION

The strain-displacement-stress-temperature relations for a plane stress case are

$$\epsilon_x = \frac{\partial u}{\partial x} = \frac{1}{E}(\sigma_x - \mu\sigma_y) + \alpha T$$

$$\epsilon_y = \frac{\partial v}{\partial y} = \frac{1}{E}(\sigma_y - \mu\sigma_x) + \alpha T$$

$$\gamma_{xy} = \frac{2(1+\mu)}{E}\tau_{xy} = \frac{\partial y}{\partial y} + \frac{\partial v}{\partial x}$$

For a simply connected region with a steady state heat flow condition the following condition is known to exist.

$$\sigma_x = \sigma_y = \tau_{xy} = 0$$

Then

$$\epsilon_x = \frac{\partial u}{\partial x} = \alpha T = \epsilon$$

$$\epsilon_y = \frac{\partial v}{\partial y} = \alpha T = \epsilon$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0$$

The rotation ω is given by

$$\omega = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Combining the equations

$$2\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0$$

yields

$$\omega = \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Then

$$\frac{\partial \omega}{\partial y} = \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial \epsilon}{\partial x}$$

and

$$\frac{\partial \omega}{\partial x} = -\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial \epsilon}{\partial y} \quad .$$

APPENDIX C

COMPUTER PROGRAMS FOR CARRYING OUT THE NUMERICAL PROCEDURE

The first program is the program used in solving for the temperature distribution. The second is the one used in solving for the special stress functions ϕ_A, ϕ_B, and ϕ_C. The third program evaluated the integrals of Michell's equations which involve the special stress function ϕ_C, and the fourth program calculates the boundary constants a, b, and c and the thermal stresses.

```
COMMENT  DETERMINATION OF THE STEADY STATE TEMPERATURE DISTRIBUTION  $
COMMENT  STUART STARRETT   ME  $
INTEGER  I, J, K, Z  $
ARRAY    T(27,16)  $
         FOR K=(1,1,150)  $
BEGIN    T(19,7)=(0.193)(T(19,8)+T(20,7)+317.5)  $
         T(19,8)=(0.50)(T(20,8)+T(19,7))  $
         T(20,3)=100.0  $
         T(20,4)=(0.0645)(T(20,5)+T(21,4)+1350.0)  $
         T(20,5)=(0.173)(T(20,6)+T(21,5)+T(20,4)+277.8)  $
         T(20,6)=(0.235)(T(20,7)+T(21,6)+T(20,5)+125.0)  $
         T(20,7)=(0.25)(T(20,8)+T(20,6)+T(21,7)+T(19,7))  $
         T(20,8)=(0.25)(T(20,9)+T(20,7)+T(21,7)+T(19,8))  $
         T(20,9)=(0.50)(T(20,8)+T(19,9))  $
         FOR I=(21,1,25)  $
BEGIN    FOR J=(3,1,I-11)  $
         T(I,J)=(0.25)(T(I,J+1)+T(I,J-1)+T(I+1,J)+T(I-1,J))  $
         T(I,2)=T(I,4)  $
         T(I,J)=T(I+1,J-1)            END          END  $
         FOR Z=(1,1,8)  $
```

```
        FOR I=(14,1,17+Z)                                             $
        FOR J=(8,1,14)                                                $
        T(I,J)=T(J+11,I-11)                                           $
        FOR I=(3,1,13)                                                $
        FOR J=(2,1,14)                                                $
        T(I,J)=T(28-I,J)                                              $
        WRITE ($$ANS,FMT)                                             $
        READ ($$DATA)                                                 $
INPUT   DATA (NONE)                                                   $
OUTPUT  ANS( FOR I=(3,1,26) $  FOR J=(2,1,15) $  T(I,J))              $
FORMAT  FMT(*5*,B2,5F15.8,W0)                                         $
        FINISH                                                        $
```

```
COMMENT   DETERMINATION OF THE STEADY STATE TEMPERATURE DISTRIBUTION   $
COMMENT   STUART STARRETT   ME                                           $
INTEGER   I, J, K, Z                                                     $
ARRAY     T(27,16)                                                       $
          FOR K=(1,1,150)                                                $
BEGIN     T(19,7)=(0.193)(T(19,8)+T(20,7)+317.5)                         $
          T(19,8)=(0.50)(T(20,8)+T(19,7))                                $
          T(20,3)=100.0                                                  $
          T(20,4)=(0.0645)(T(20,5)+T(21,4)+1350.0)                       $
          T(20,5)=(0.173)(T(20,6)+T(21,5)+T(20,4)+277.8)                 $
          T(20,6)=(0.235)(T(20,7)+T(21,6)+T(20,5)+125.0)                 $
          T(20,7)=(0.25)(T(20,8)+T(20,6)+T(21,7)+T(19,7))                $
          T(20,8)=(0.25)(T(20,9)+T(20,7)+T(21,7)+T(19,8))                $
          T(20,9)=(0.50)(T(20,8)+T(19,9))                                $
          FOR I=(21,1,25)                                                $
BEGIN     FOR J=(3,1,I-11)                                               $
          T(I,J)=(0.25)(T(I,J+1)+T(I,J-1)+T(I+1,J)+T(I-1,J))             $
          T(I,2)=T(I,4)                                                  $
          T(I,J)=T(I+1,J-1)                 END              END         $
          FOR Z=(1,1,8)                                                  $
          FOR I=(14,1,17+Z)                                              $
          FOR J=(8,1,14)                                                 $
          T(I,J)=T(J+11,I-11)                                            $
          FOR I=(3,1,13)                                                 $
          FOR J=(2,1,14)                                                 $
          T(I,J)=T(28-I,J)                                               $
          WRITE ($$ANS,FMT)                                              $
          READ ($$DATA)                                                  $
INPUT     DATA (NONE)                                                    $
OUTPUT    ANS( FOR I=(3,1,26) $  FOR J=(2,1,15) $  T(I,J))               $
FORMAT    FMT(*5*,B2,5F15.8,W0)                                          $
          FINISH                                                         $
```

```
COMMENT   DETERMINATION OF THE STRESS FUNCTION PHE AND THE
          EVALUATION OF THE THERMAL STRESSES                                  $
COMMENT   STUART STARRETT   ME                                                $
          INTEGER I,J                                                         $
ARRAY     PHEA(27,16), PHEB(27,16), PHEC(27,16), PHE(27,16),
          SIGMAX(27,16), SIGMAY(27,16), TAUXY(27,16)                          $
          EALPHA=(30.0)(6.50)                                                 $
          INGRL1T=-513.06888                                                  $
          INGRL2T=0.000                                                       $
          INGRL3T=0.000                                                       $
          INGRL1PHEA=0.000                                                    $
          INGRL2PHEA=0.000                                                    $
          INGRL3PHEA=-5.8947916                                               $
          INGRL1PHEB=0.000                                                    $
          INGRL2PHEB=-5.8946088                                               $
          INGRL3PHEB=0.000                                                    $
          INGRL1PHEC=-27.384760                                               $
          INGRL2PHEC=0.000                                                    $
          INGRL3PHEC=0.000                                                    $
          C=-(EALPHA)(INGRL1T)/(INGRL1PHEC)                                   $
          B=-(EALPHA)(INGRL2T)/(INGRL2PHEB)                                   $
          A=-(EALPHA)((INGRL3T)-(C)(INGRL3PHEC))/(INGRL3PHEA)                 $
          READ($$DATA1)                                                       $
          READ($$DATA2)                                                       $
          READ($$DATA3)                                                       $
INPUT     DATA1( FOR I=(1,1,27) $  FOR J=(1,1,16) $  PHEA(I,J))               $
INPUT     DATA2( FOR I=(1,1,27) $  FOR J=(1,1,16) $  PHEB(I,J))               $
INPUT     DATA3( FOR I=(1,1,27) $  FOR J=(1,1,16) $  PHEC(I,J))               $
          FOR I=(1,1,27)                                                      $
          FOR J=(1,1,16)                                                      $
          PHE(I,J)=A.PHEA(I,J)+B.PHEB(I,J)+C.PHEC(I,J)                        $
          FOR I=(2,1,26)                                                      $
          FOR J=(3,1,15)                                                      $
BEGIN     SIGMAX(I,J)=(A)(PHEA(I,J+1)-2.PHEA(I,J)+PHEA(I,J-1))/(0.0625)
                      (B)(PHEB(I,J+1)-2.PHEB(I,J)+PHEB(I,J-1))/(0.0625)
                      (C)(PHEC(I,J+1)-2.PHEC(I,J)+PHEC(I,J-1))/(0.0625)$
          SIGMAY(I,J)=(A)(PHEA(I+1,J)-2.PHEA(I,J)+PHEA(I-1,J))/(0.0625)
                      (B)(PHEB(I+1,J)-2.PHEB(I,J)+PHEB(I-1,J))/(0.0625)
                      (C)(PHEC(I+1,J)-2.PHEC(I,J)+PHEC(I-1,J))/(0.0625)$
```

```
         TAUXY(I,J)=-(A)(PHEA(I+1,J+1)+PHEA(I-1,J-1)-PHEA(I+1,J-1)-
                        PHEA(I-1,J+1))/((4.0)(0.0625))
                 -(B)(PHEB(I+1,J+1)+PHEB(I-1,J-1)-PHEB(I+1,J-1)-
                        PHEB(I-1,J+1))/((4.0)(0.0625))
                 -(C)(PHEC(I+1,J+1)+PHEC(I-1,J-1)-PHEC(I+1,J-1)-
                        PHEC(I-1,J+1))/((4.0)(0.0625))       END       $
         WRITE($$TITLE1)                                               $
         WRITE($$ANS1,FMT1)                                            $
         WRITE($$TITLE2)                                               $
         FOR I=(1,1,27)                                                $
         FOR J=(3,1,16)                                                $
         WRITE($$ANS2,FMT2)                                            $
OUTPUT   ANS1(A, B, C)                                                 $
OUTPUT   ANS2(I, J, PHE(I,J), SIGMAX(I,J), SIGMAY(I,J), TAUXY(I,J))    $
FORMAT   TITLE1(B10,*STRESS FUNCTION CONSTANTS*, W4)                   $
FORMAT   TITLE2(B7,*I*,B7,*J*,B19,*PHE(I,J)*,B13,*SIGMAX(I,J)*,
              B8,*SIGMAY(I,J)*,B8,*TAUXY(I,J)*,W4)                     $
FORMAT   FMT1(B10,*A=*,S14.8,W4,(B10,*B=*,S14.8,W4,
            (B10,*C=*,S14.8,W4)))                                      $
FORMAT   FMT2(B6,I2,B6,I2,B8,4S20.8,W4)                                $
         FINISH                                                        $
```

```
COMMENT   EVALUATION OF THE LINE INTEGRALS INVOLVING THE LAPLACIAN
          OF PHEC                                                          $
COMMENT   STUART STARRET    ME                                             $
PROCEDURE SIMPS(N,DELX,F())                                                $
BEGIN     INTEGER I,N                                                      $
          S=0.0                                                            $
          S=F(1)+F(N+1)                                                    $
          FOR I=(2,2,N)                                                    $
          S=S+(4.0)(F(I))                                                  $
          FOR I=(3,2,N-1)                                                  $
          S=S+(2.0)(F(I))                                                  $
          SIMPS()=(DELX/3.0)(S)                                            $
          RETURN                                           END SIMPS()     $
          INTEGER I,J                                                      $
ARRAY     PHEC(27,16), D2PHEC(25,14), XDD2PHEC(25,14), YDD2PHEC(25,14),
          X1(9), X2(9), X3(9), X4(9), Z1(9), Z2(9), Z3(9), Z4(9),
          A1(11), A2(11), A3(11), A4(11)                                   $
          READ($$DATA)                                                     $
INPUT     DATA(FOR I=(1,1,27) $  FOR J=(1,1,16) $  PHEC(I,J))              $
          FOR I=(3,1,25)                                                   $
          FOR J=(2,1,14)                                                   $
          D2PHEC(I,J)=(PHEC(I+1,J)+PHEC(I-1,J)+PHEC(I,J+1)+PHEC(I,J-1)
                    -4.PHEC(I,J))/(0.0625)                                 $
          FOR I=(4,1,6)                                                    $
          FOR J=(2,1,13)                                                   $
BEGIN     XDD2PHEC(I,J)=(D2PHEC(I-1,J)-D2PHEC(I+1,J))/(0.50)               $
          YDD2PHEC(I,J)=(D2PHEC(I,J-1)-D2PHEC(I,J+1))/(0.50) END           $
          FOR I=(4,1,24)                                                   $
          FOR J=(11,1,13)                                                  $
BEGIN     XDD2PHEC(I,J)=(D2PHEC(I-1,J)-D2PHEC(I+1,J))/(0.50)               $
          YDD2PHEC(I,J)=(D2PHEC(I,J+1)-D2PHEC(I,J-1))/(0.50) END           $
          FOR I=(22,1,24)                                                  $
          FOR J=(2,1,13)                                                   $
BEGIN     XDD2PHEC(I,J)=(D2PHEC(I+1,J)-D2PHEC(I-1,J))/(0.50)               $
          YDD2PHEC(I,J)=(D2PHEC(I,J+1)-D2PHEC(I,J-1))/(0.50) END           $
          FOR J=(3,1,11)                                                   $
BEGIN     X1(J-2)=XDD2PHEC(22,J)                                           $
          X4(J-2)=XDD2PHEC(6,J)                                            $
          Z1(J-2)=(J-3)(0.25)(YDD2PHEC(22,J))+(2.0)(XDD2PHEC(22,J))        $
```

```
        Z4(J-2)=(J-3)(0.25)(YDD2PHEC(6,J))-(2.0)(XDD2PHEC(6,J))   END $
        FOR J=(3,1,13)                                                $
BEGIN   A1(J-2)=XDD2PHEC(24,J)                                        $
        A4(J-2)=XDD2PHEC(4,J)                              END        $
        FOR I=(14,1,22)                                               $
BEGIN   X2(I-13)=YDD2PHEC(I,11)                                       $
        Z2(I-13)=(2.0)(XDD2PHEC(I,11))+(I-14)(0.25)(YDD2PHEC(I,11))
                                                              END     $
        FOR I=(14,1,24)                                               $
        A2(J-13)=YDD2PHEC(I,13)                                       $
        FOR I=(6,1,14)                                                $
BEGIN   X3(I-5)=YDD2PHEC(I,11)                                        $
        Z3(I-5)=(2.0)(XDD2PHEC(I,11))-(14-I)(0.25)(YDD2PHEC(I,11))
                                                              END     $
        FOR I=(4,1,14)                                                $
        A3(J-3)=YDD2PHEC(I,13)                                        $
        INGRL1PHEC=(2.0)(SIMPS(8,0.25,X1())+SIMPS(8,0.25,X2())
                        SIMPS(8,0.25,X3())+SIMPS(8,0.25,X4()))        $
        INGRL2PHEC=0.00                                               $
        INGRL3PHEC=(2.0)(SIMPS(8,0.25,Z1())+SIMPS(8,0.25,Z2())
                        SIMPS(8,0.25,Z3())+SIMPS(8,0.25,Z4()))        $
        INGRL4PHEC=(2.0)(SIMPS(10,0.25,A1())+SIMPS(10,0.25,A2())
                        SIMPS(10,0.25,A3())+SIMPS(10,0.25,A4()))      $
        WRITE($$ANS,FMT)                                              $
OUTPUT  ANS(INGRL1PHEC, INGRL2PHEC, INGRL3PHEC, INGRL4PHEC)           $
FORMAT  FMT(*INGRL1PHEC=*,F14.8,W4,(*INGRL2PHEC=*,F14.8,W4,
           (*INGRL3PHEC=*,F14.8,W4,(*INGRL4PHEC=*,F14.8,W4))))        $
        FINISH                                                        $
```

```
COMMENT   DETERMINATION OF THE SPECIAL STRESS FUNCTIONS PHEA, PHEB,
          AND PHEC BY THE GAUSS SIEDEL RELAXATION METHOD              $
COMMENT   STUART STARRETT  ME                                         $
INTEGER   I,J,K,N                                                     $
ARRAY     PHEA(27,16), PHEB(27,16), A1(5), B1(5), C1(5), D1(5), E1(5),
          F1(5), G1(5), B(5), C(5), F(5), G(5),PHEC(27,16)            $
          A1(1)=0.08                                                  $
          A1(2)=0.36                                                  $
          A1(3)=0.80                                                  $
          A1(4)=0.72                                                  $
          A1(5)=0.56                                                  $
          FOR N=(1,1,5)                                               $
BEGIN     B1(N)=((2+A1(N))/((1+A1(N))(1+A1(N))))                      $
          C1(N)=((1+A1(N))/((2+A1(N))(2+A1(N))))                      $
          D1(N)=((3+2(A1(N)))/((1+A1(N))(2+A1(N))))                   $
          E1(N)=(B1(N))/(2+A1(N))                                     $
          F1(N)=(C1(N))/(1+A1(N))                                     $
          G1(N)=(D1(N))/(3+2(A1(N)))                      END         $
          FOR N=(1,1,5)                                               $
BEGIN     B(N)=(1-A1(N))(1-A1(N))(B1(N)+(1-A1(N))(E1(N)))             $
          C(N)=(1-A1(N))(1-A1(N))(C1(N)+(1-A1(N))(F1(N)))             $
          F(N)=(A1(N))(A1(N))(B1(N)-(A1(N))(E1(N)))                   $
          G(N)=(A1(N))(A1(N))(C1(N)-(A1(N))(F1(N)))       END         $
          READ($$DATA1)                                               $
          READ($$DATA3)                                               $
INPUT     DATA1( FOR I=(1,1,27) $  FOR J=(1,1,16) $  PHEA(I,J))       $
INPUT     DATA3( FOR I=(1,1,27) $  FOR J=(1,1,16) $  PHEC(I,J))       $
          FOR K=(1,1,200)                                             $
BEGIN     FOR I=(14,1,26)                                             $
BEGIN     PHEA(I,15)=(I-14)(0.25)                                     $
          PHEA(I,16)=PHEA(I,14)                                       $
          PHEA(I,1)=PHEA(I,5)                                         $
          PHEA(I,2)=PHEA(I,4)                             END         $
          FOR J=(3,1,15)                                              $
BEGIN     PHEA(26,J)=3.00                                             $
          PHEA(27,J)=PHEA(26,J)+0.25                                  $
          PHEA(12,J)=-(PHEA(16,J))                                    $
          PHEA(13,J)=-(PHEA(15,J))                        END         $
          PHEA(19,3)=PHEA(21,3)                                       $
```

```
PHEA(14,8)=PHEA(14,10)                                              $
FOR I=(14,1,20)                                                     $
FOR J=(10,1,14)                                                     $
PHEA(I,J)=0.05(8(PHEA(I+1,J)+PHEA(I-1,J)+PHEA(I,J+1)
          PHEA(I,J-1))-2(PHEA(I+1,J+1)+PHEA(I+1,J-1)
          PHEA(I-1,J+1)+PHEA(I-1,J-1))-(PHEA(I,J-2)
          PHEA(I-2,J)+PHEA(I,J+2)+PHEA(I+2,J)))                     $
FOR I=(21,1,25)                                                     $
FOR J=(3,1,14)                                                      $
PHEA(I,J)=0.05(8(PHEA(I+1,J)+PHEA(I-1,J)+PHEA(I,J+1)
          PHEA(I,J-1))-2(PHEA(I+1,J+1)+PHEA(I+1,J-1)
          PHEA(I-1,J+1)+PHEA(I-1,J-1))-(PHEA(I,J-2)
          PHEA(I-2,J)+PHEA(I,J+2)+PHEA(I+2,J)))                     $
FOR I=(20,1,20)                                                     $
FOR J=(7,1,9)                                                       $
PHEA(I,J)=0.05(8(PHEA(I+1,J)+PHEA(I-1,J)+PHEA(I,J+1)
          PHEA(I,J-1))-2(PHEA(I+1,J+1)+PHEA(I+1,J-1)
          PHEA(I-1,J+1)+PHEA(I-1,J-1))-(PHEA(I,J-2)
          PHEA(I-2,J)+PHEA(I,J+2)+PHEA(I+2,J)))                     $
FOR I=(18,1,20)                                                     $
FOR J=(9,1,9)                                                       $
PHEA(I,J)=0.05(8(PHEA(I+1,J)+PHEA(I-1,J)+PHEA(I,J+1)
          PHEA(I,J-1))-2(PHEA(I+1,J+1)+PHEA(I+1,J-1)
          PHEA(I-1,J+1)+PHEA(I-1,J-1))-(PHEA(I,J-2)
          PHEA(I-2,J)+PHEA(I,J+2)+PHEA(I+2,J)))                     $
PHEA(19,8)=0.05(8(PHEA(19,9)+PHEA(20,8))-2(PHEA(18,9)+
 PHEA(20,9)+PHEA(20,7))-(PHEA(19,10)+PHEA(21,8)))                   $
PHEA(20,4)=(F(1).PHEA(21,4)+G(1).PHEA(22,4))                        $
PHEA(19,4)=(B(1).PHEA(21,4)+C(1).PHEA(22,4))                        $
PHEA(20,5)=(F(2).PHEA(21,5)+G(2).PHEA(22,5))                        $
PHEA(19,5)=(B(2).PHEA(21,5)+C(2).PHEA(22,5))                        $
PHEA(20,6)=(F(3).PHEA(21,6)+G(3).PHEA(22,6))                        $
PHEA(19,6)=0.50((B(3).PHEA(21,6)+C(3).PHEA(22,6))+
              (B(4).PHEA(19,8)+C(4).PHEA(19,9)))                    $
PHEA(19,7)=0.50((F(4).PHEA(19,8)+G(4).PHEA(19,9))+
              (F(5).PHEA(20,7)+G(5).PHEA(21,7)))                    $
PHEA(18,7)=0.50((B(5).PHEA(20,7)+C(5).PHEA(21,7))+
              (B(5).PHEA(18,9)+C(5).PHEA(18,10)))                   $
PHEA(18,8)=0.50((F(5).PHEA(18,9)+G(5).PHEA(18,10))+
```

```
                    (F(4).PHEA(19,8)+G(4).PHEA(20,8)))               $
        PHEA(17,8)=0.50((B(4).PHEA(19,8)+C(4).PHEA(20,8))+
                    (B(3).PHEA(17,10)+C(3).PHEA(17,11)))             $
        PHEA(17,9)=(F(3).PHEA(17,10)+G(3).PHEA(17,11))               $
        PHEA(16,8)=(B(2).PHEA(16,10)+C(2).PHEA(16,11))               $
        PHEA(16,9)=(F(2).PHEA(16,10)+G(2).PHEA(16,11))               $
        PHEA(15,8)=(B(1).PHEA(15,10)+C(1).PHEA(15,11))               $
        PHEA(15,9)=(F(1).PHEA(15,10)+G(1).PHEA(15,11))    END        $
        FOR I=(1,1,14)                                               $
        FOR J=(1,1,16)                                               $
        PHEA(I,J)=-(PHEA(28-I,J))                                    $
        FOR I=(14,1,27)                                              $
        FOR J=(3,1,16)                                               $
        PHEB(I,J)=PHEA(J+11,I-11)                                    $
        FOR I=(1,1,13)                                               $
        FOR J=(3,1,16)                                               $
        PHEB(I,J)=PHEB(28-I,J)                                       $
        FOR I=(1,1,27)                                               $
BEGIN   PHEB(I,1)=-(PHEB(I,5))                                       $
        PHEB(I,2)=-(PHEB(I,4))                            END        $
        FOR K=(1,1,500)                                              $
BEGIN   FOR I=(14,1,26)                                              $
BEGIN   PHEC(I,15)=1.00                                              $
        PHEC(I,16)=PHEC(I,14)                                        $
        PHEC(I,1)=PHEC(I,5)                                          $
        PHEC(I,2)=PHEC(I,4)                               END        $
        FOR J=(3,1,15)                                               $
BEGIN   PHEC(26,J)=1.00                                              $
        PHEC(27,J)=PHEC(25,J)                                        $
        PHEC(12,J)=PHEC(16,J)                                        $
        PHEC(13,J)=PHEC(15,J)                             END        $
        PHEC(19,3)=PHEC(21,3)                                        $
        PHEC(14,8)=PHEC(14,10)                                       $
        FOR I=(14,1,20)                                              $
        FOR J=(10,1,14)                                              $
        PHEC(I,J)=0.05(8(PHEC(I+1,J)+PHEC(I-1,J)+PHEC(I,J+1)
                  PHEC(I,J-1))-2(PHEC(I+1,J+1)+PHEC(I+1,J-1)
                  PHEC(I-1,J+1)+PHEC(I-1,J-1))-(PHEC(I,J-2)
                  PHEC(I-2,J)+PHEC(I,J+2)+PHEC(I+2,J)))              $
```

```
FOR I=(21,1,25)                                                   $
FOR J=(3,1,14)                                                    $
PHEC(I,J)=0.05(8(PHEC(I+1,J)+PHEC(I-1,J)+PHEC(I,J+1)
          PHEC(I,J-1))-2(PHEC(I+1,J+1)+PHEC(I+1,J-1)
          PHEC(I-1,J+1)+PHEC(I-1,J-1))-(PHEC(I,J-2)
          PHEC(I-2,J)+PHEC(I,J+2)+PHEC(I+2,J)))                   $
FOR I=(20,1,20)                                                   $
FOR J=(7,1,9)                                                     $
PHEC(I,J)=0.05(8(PHEC(I+1,J)+PHEC(I-1,J)+PHEC(I,J+1)
          PHEC(I,J-1))-2(PHEC(I+1,J+1)+PHEC(I+1,J-1)
          PHEC(I-1,J+1)+PHEC(I-1,J-1))-(PHEC(I,J-2)
          PHEC(I-2,J)+PHEC(I,J+2)+PHEC(I+2,J)))                   $
FOR I=(18,1,20)                                                   $
FOR J=(9,1,9)                                                     $
PHEC(I,J)=0.05(8(PHEC(I+1,J)+PHEC(I-1,J)+PHEC(I,J+1)
          PHEC(I,J-1))-2(PHEC(I+1,J+1)+PHEC(I+1,J-1)
          PHEC(I-1,J+1)+PHEC(I-1,J-1))-(PHEC(I,J-2)
          PHEC(I-2,J)+PHEC(I,J+2)+PHEC(I+2,J)))                   $
PHEC(19,8)=0.05(8(PHEC(19,9)+PHEC(20,8))-2(PHEC(18,9)+
 PHEC(20,9)+PHEC(20,7))-(PHEC(19,10)+PHEC(21,8)))                 $
PHEC(20,4)=(F(1).PHEC(21,4)+G(1).PHEC(22,4))                      $
PHEC(19,4)=(B(1).PHEC(21,4)+C(1).PHEC(22,4))                      $
PHEC(20,5)=(F(2).PHEC(21,5)+G(2).PHEC(22,5))                      $
PHEC(19,5)=(B(2).PHEC(21,5)+C(2).PHEC(22,5))                      $
PHEC(20,6)=(F(3).PHEC(21,6)+G(3).PHEC(22,6))                      $
PHEC(19,6)=0.50((B(3).PHEC(21,6)+C(3).PHEC(22,6))+
              (B(4).PHEC(19,8)+C(4).PHEC(19,9)))                  $
PHEC(19,7)=0.50((F(4).PHEC(19,8)+G(4).PHEC(19,9))+
              (F(5).PHEC(20,7)+G(5).PHEC(21,7)))                  $
PHEC(18,7)=0.50((B(5).PHEC(20,7)+C(5).PHEC(21,7))+
              (B(5).PHEC(18,9)+C(5).PHEC(18,10)))                 $
PHEC(18,8)=0.50((F(5).PHEC(18,9)+G(5).PHEC(18,10))+
              (F(4).PHEC(19,8)+G(4).PHEC(20,8)))                  $
PHEC(17,8)=0.50((B(4).PHEC(19,8)+C(4).PHEC(20,8))+
              (B(3).PHEC(17,10)+C(3).PHEC(17,11)))                $
PHEC(17,9)=(F(3).PHEC(17,10)+G(3).PHEC(17,11))                    $
PHEC(16,8)=(B(2).PHEC(16,10)+C(2).PHEC(16,11))                    $
PHEC(16,9)=(F(2).PHEC(16,10)+G(2).PHEC(16,11))                    $
PHEC(15,8)=(B(1).PHEC(15,10)+C(1).PHEC(15,11))                    $
```

```
        PHEC(15,9)=(F(1).PHEC(15,10)+G(1).PHEC(15,11))    END               $
        FOR I=(1,1,14)                                                      $
        FOR J=(1,1,16)                                                      $
        PHEC(I,J)=PHEC(28-I,J)                                              $
        WRITE($$ANS1,FMT)                                                   $
        WRITE($$ANS2,FMT)                                                   $
        WRITE($$ANS3,FMT)                                                   $
OUTPUT  ANS1( FOR I=(1,1,27) $  FOR J=(1,1,16) $  PHEA(I,J))                $
OUTPUT  ANS2( FOR I=(1,1,27) $  FOR J=(1,1,16) $  PHEB(I,J))                $
OUTPUT  ANS3( FOR I=(1,1,27) $  FOR J=(1,1,16) $  PHEC(I,J))                $
FORMAT  FMT(*5*,B2,5F15.8,W0)                                               $
        FINISH                                                              $
```

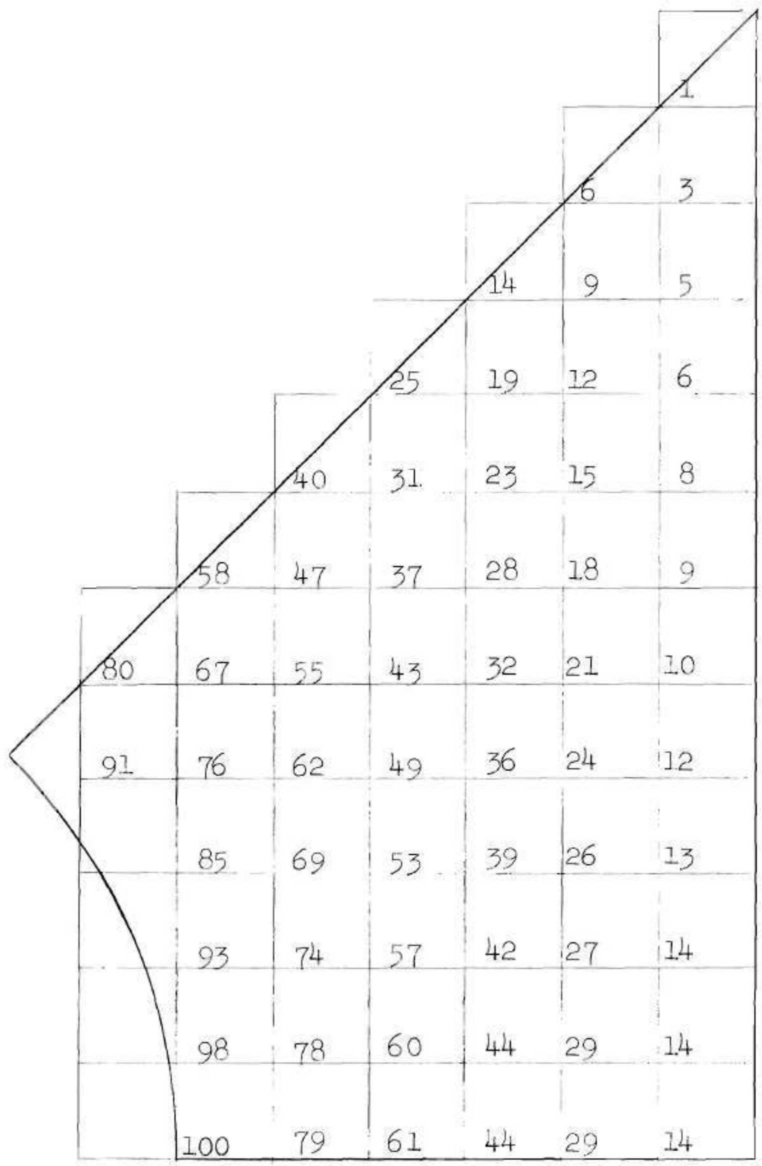


Figure 15. Temperature Distribution (° F).

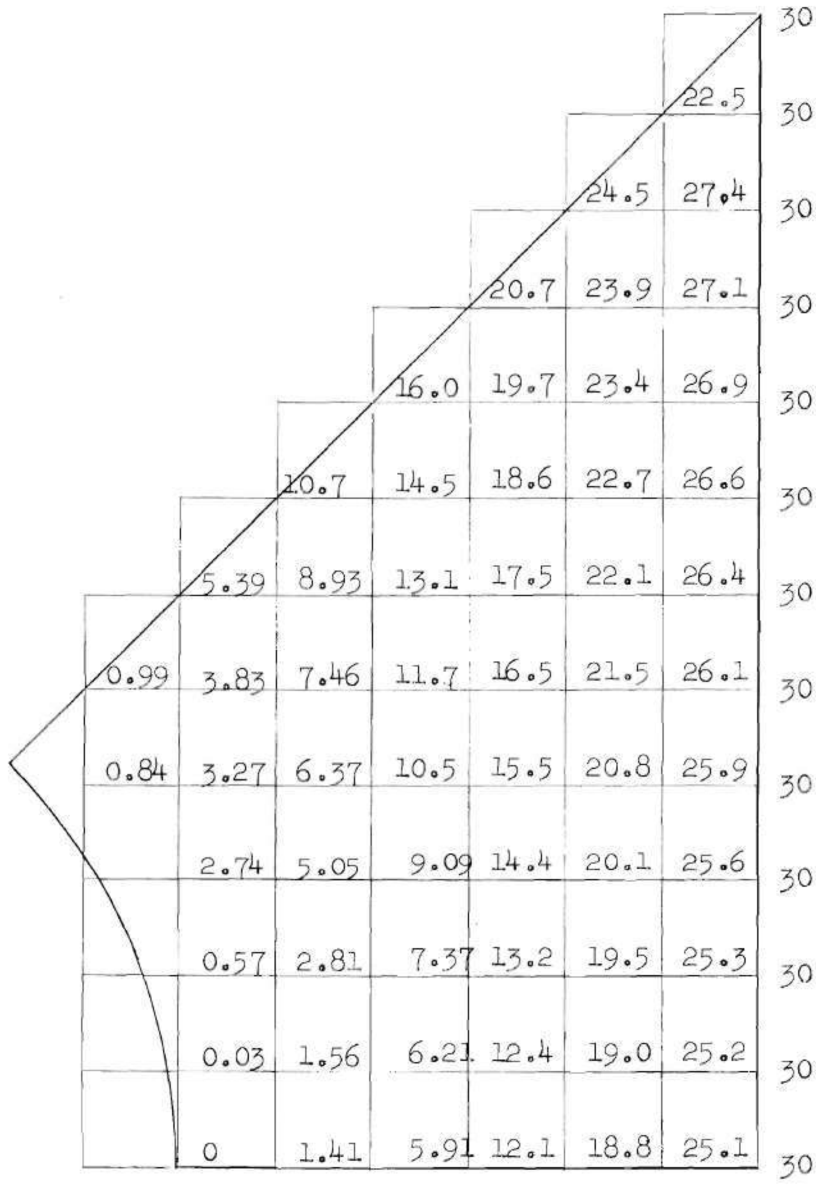


Figure 16. Results of Relaxation Calculation for ϕ_A. (ϕ_A x 10)

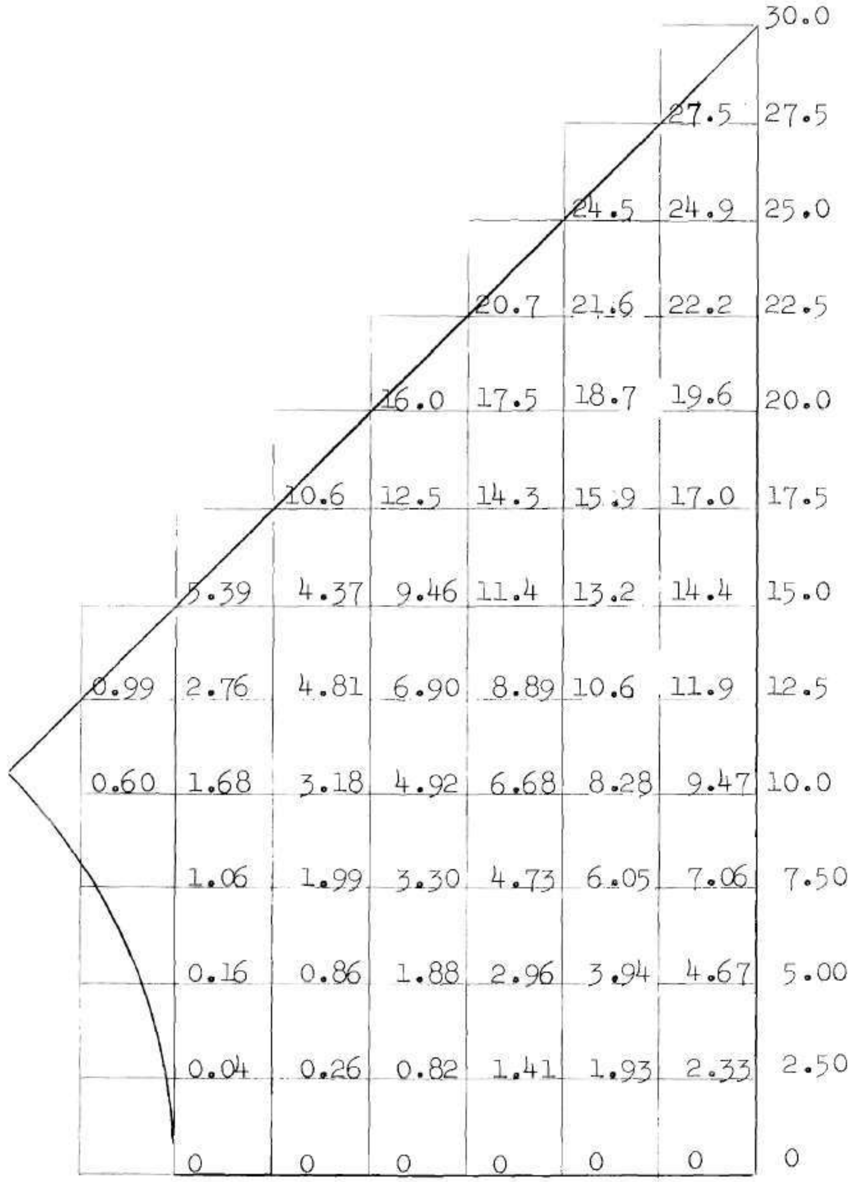


Figure 17. Results of the Relaxation Calculation for ϕ_B. (ϕ_B x 10)

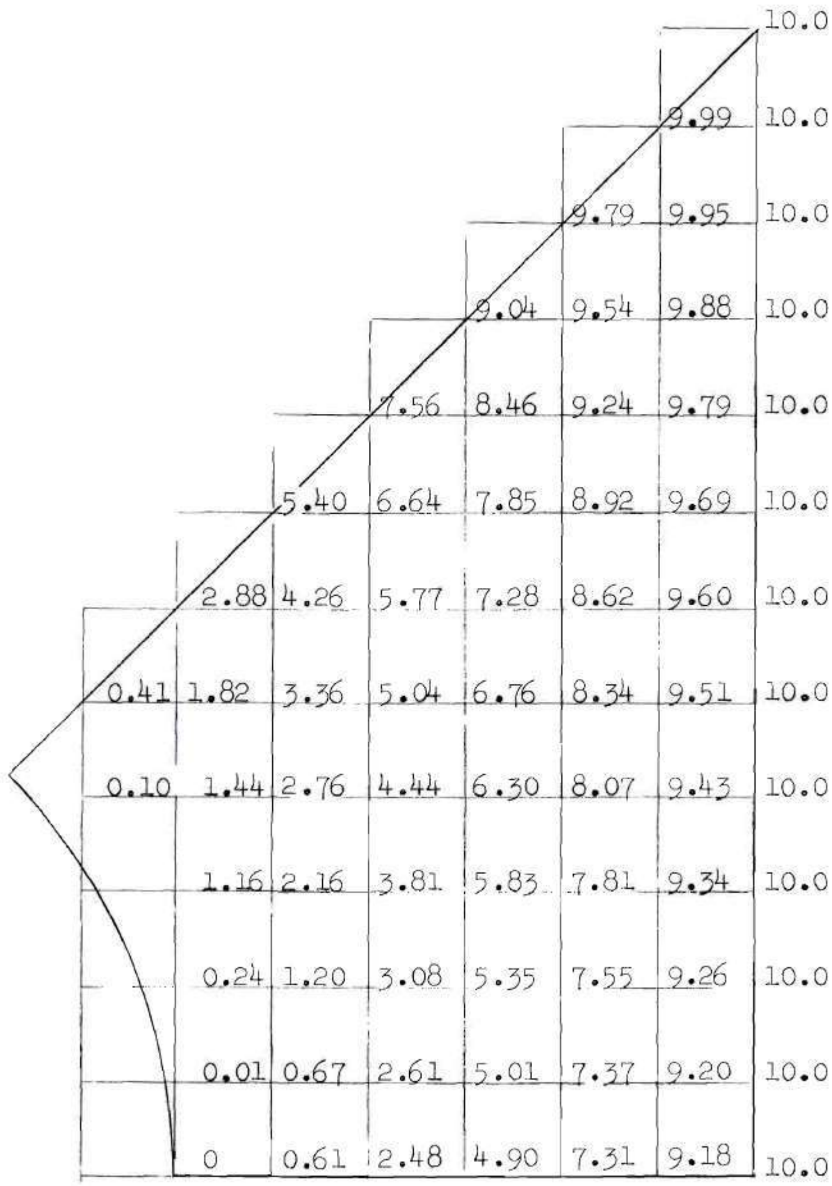


Figure 18. Results of the Relaxation Calculation for ϕ_C. ($\phi_C \times 10$)

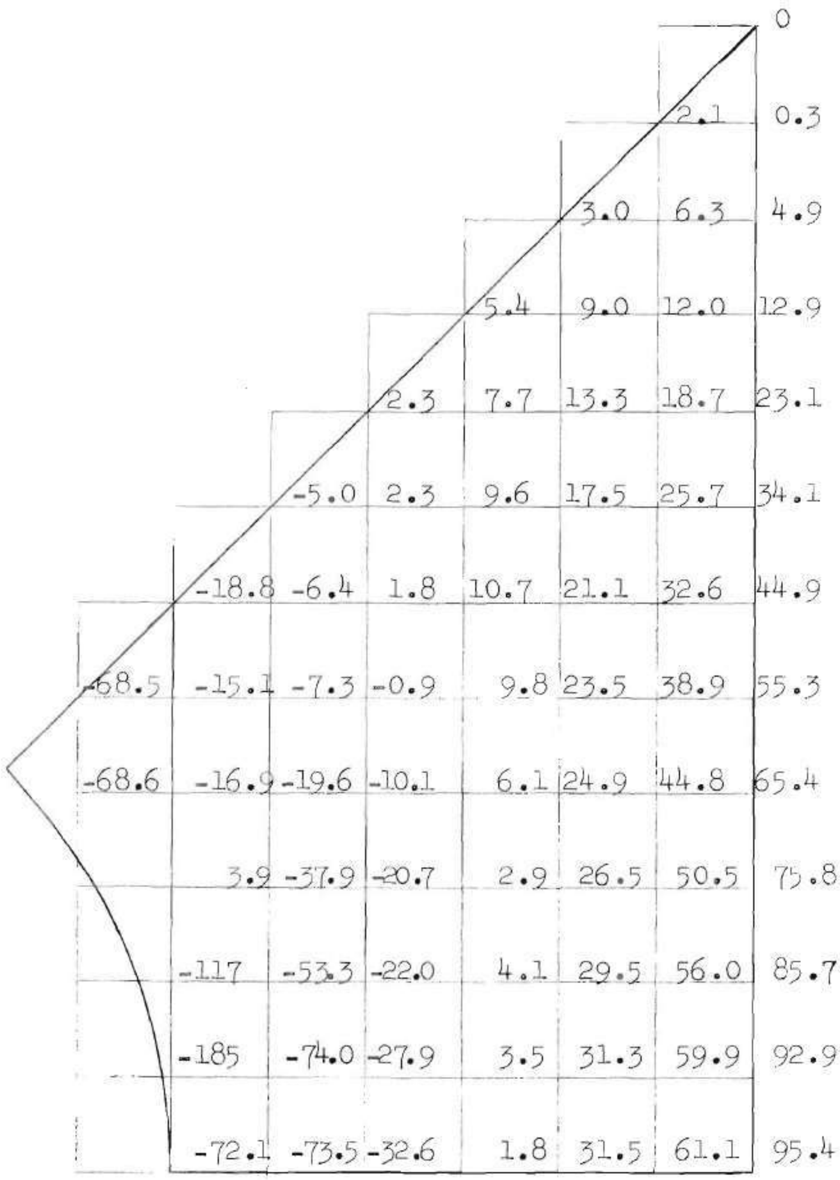


Figure 19. Y-Stress Distribution ($\sigma_y \times 10^{-2}$).

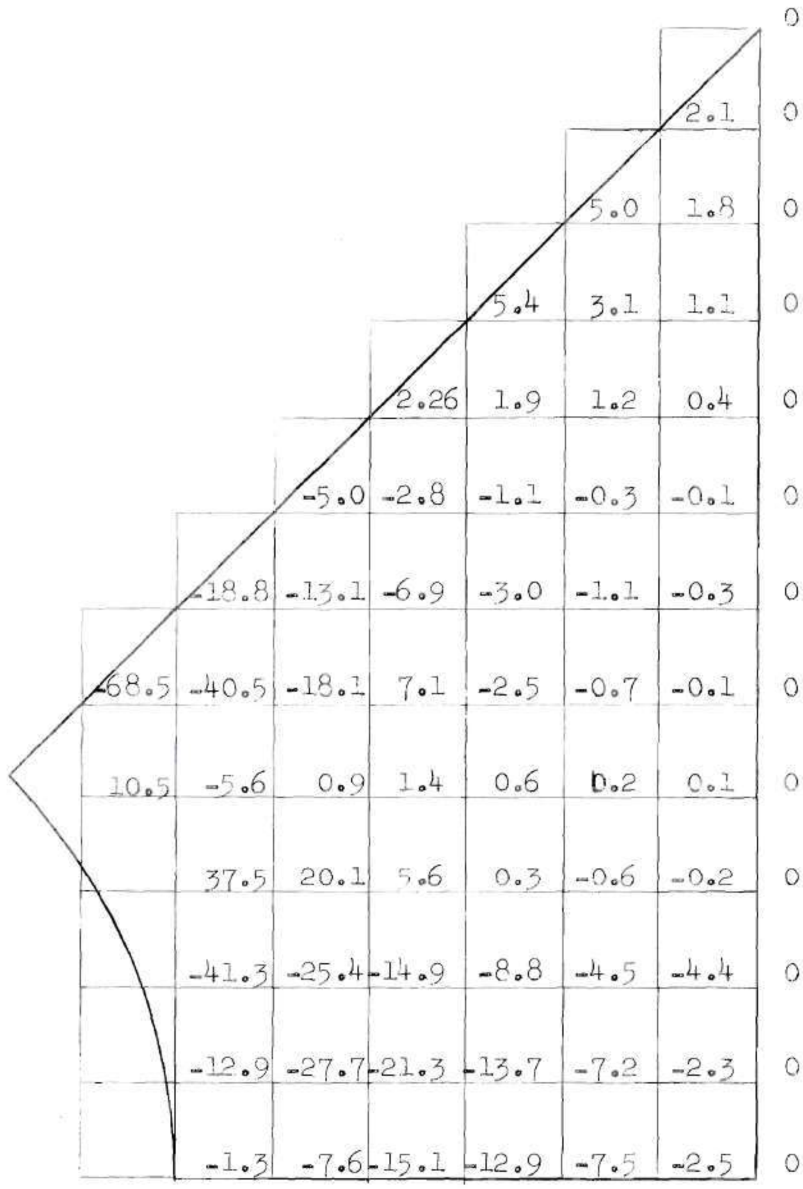


Figure 20. X-Stress Distribution ($\sigma_x \times 10^{-2}$).

BIBLIOGRAPHY

1. Michell, J. H., "On the Direct Determination of Stress in an Elastic Solic, with Application to the Theory of Plates," Proceedings of the London Mathematics Society, 31, 1900, pp. 100-124.

2. Prager, W., "On Plane Elastic Strain in Doubly-Connected Domains," Quarterly of Applied Mathematics, 3, No. 4, January 1946, pp. 377-380.

3. Southwell, R. V., Theory of Elasticity, Oxford University Press, Oxford, 1936, pp. 368-371.

4. Mindlin, R. D., and Salvadori, M. G., "Analogies," Handbook of Experimental Stress Analysis, John Wiley and Sons, New York, 1950.

5. Biot, M. A., "A General Property of Two Dimensional Thermal Stress Distribution," Philosophical Magazine, 19, 1935, pp. 540-549.

6. Weibel, E. E., "Thermal Stresses in Cylinders by the Photoelastic Method," Proceedings of the Fifth International Congress Applied Mechanics, Cambridge, Mass., 1938, pp. 213-220.

7. Dolan, T. J., and Murray, W. M., "Photoelasticity," Handbook of Experimental Stress Analysis, John Wiley and Sons, New York, 1950.

8. Filon, L. N. G., "A Discussion of the Theory of Dislocations in the Case of Two-Dimensional Systems," British Association Reports, 1921, p. 305.

9. Love, A. E. H., The Mathematical Theory of Elasticity, Fourth Edition, Dover Publications, New York, 1944.

10. Timoshenko, S., and Goodies, J. N., Theory of Elasticity, McGraw-Hill Book Co., Inc., New York, 1951.

11. Frocht, M. M., Photoelasticity, 1, John Wiley and Sons, New York, 1941.

12. Holms, A. G., "A Biharmonic Relaxation Method for Calculating Thermal Stress in Cooled Irregular Cylinders," NACA TN 2434, August, 1951.

13. Kreith, F., Principles of Heat Transfer, International Textbook Co., Scranton, 1960.

14. Shaw, F. S., Relaxation Methods, Dover Publications.